Symplectic Mappings of Sobolev Class

Inauguraldissertation

der Philosophisch-naturwissenschaftlichen Fakultät

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Leiter der Arbeit: Prof. Dr. H. M. Reimann
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Preface

Symplectic mappings are mostly studied from the global viewpoint of symplectic topology. This thesis makes a different approach to the understanding of symplectic mappings. The focus here lies mainly in the local analytical behaviour, especially in regularity properties. This approach has been motivated by results in the lively theory of quasiconformal mappings. In particular, the work of Tadeusz Iwaniec and others has given new impetus to the study of regularity questions. Symplectic geometry is one further subject of interest of my advisor Prof. Hans Martin Reimann. This combination prompted the question whether techniques from the quasiconformal theory could be imitated for the symplectic theory. In this thesis it is shown that non-differentiable symplectic mappings in Sobolev spaces behave nicely in various respects; though it was not possible to build a counterpart of the quasiconformal theory.

In the first chapter some minimization problems are solved within the symplectic class. As symplectic mappings are of so-called finite dilatation they are good enough that the global invertibility property in chapter 2 follows. Remarkably, all the coordinate functions of a quasiconformal symplectic mapping satisfy the same Beltrami equation; here the techniques of complex analysis in several variables come into play. This is presented in chapter 3. Finally, the last chapter collects some calculations on exterior algebras. The consideration of differential forms was one innovation in the quasiconformal theory.

I would like to express me deep gratitude to Prof. Hans Martin Reimann for his constant encouraging and patient support during the last years. He was an incessant source of ideas during this work. In this time, I met many people who were willing to share their knowledge with me. To name just a few, I thank Tadeusz Iwaniec and Serguei Vodopyanov for explaining me their work. A special thanks goes to Zoltán Balogh for his guidance and numerous scientific discussions we had. Finally I thank F. W. and all my friends for their support, be it mathematical or moral.

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Andreas Bieri

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1 Symplectic Mappings and the Calculus of Variations

1.1 Symplectically-Harmonic Mappings

1.1.1 Introduction

The close relationship between holomorphic functions $\mathbb{C} \to \mathbb{C}$ and harmonic functions is a special feature of the one-dimensional function theory. For higher dimensional complex manifolds, these concepts are less related in general. However, in the case of Kähler manifolds, the Kähler condition expresses a compatibility between the Riemannian and the complex structure which connects the holomorphic and harmonic theory. The following theorem is a typical statement in this setting:

Theorem 1.1 *([8, p.51])*

1. *If φ is a holomorphic or anti-holomorphic mapping between Kähler manifolds, then it is a harmonic mapping and minimizes the energy in its homotopy class.*
2. *If φ minimizes the energy in its homotopy class and is homotopic to a holomorphic or anti-holomorphic mapping, then φ is holomorphic or anti-holomorphic.*

This theorem is one reason to believe that the class of symplectic energy-minimizing mappings defined below might be a good substitute for biholomorphic mappings.

Notations for this chapter: Let (M, g) and (N, h) denote complex manifolds of real dimension $2n$, carrying Riemannian metrics g and h. The respective complex structures will be denoted by J^g and J^h. These are endomorphisms $J^g_x : T_xM \to T_xM$, $J^h_y : T_yN \to T_yN$, such that $(J^g_x)^2 = -id_{T_xM}$ and $(J^h_y)^2 = -id_{T_yN}$. The Riemannian metrics are *Hermitian* or *J-invariant* if $g(J^gX, J^gY) = g(X, Y)$, $h(J^hX, J^hY) = h(X, Y)$ for all vector fields X, Y of the respective tangent spaces. Together with the complex structures we can then define non-degenerate 2-forms $\omega_1(X, Y) := g(X, J^gY)$ and $\omega_2(X, Y) := h(X, J^hY)$, the *Kähler-forms* of (M, g, J^g) and (N, h, J^h). If these 2-forms are also closed they are symplectic forms and the manifolds are called *Kähler manifolds*. Without further notice, the manifolds (M, g, J^g, ω_1) and (N, h, J^h, ω_2) are supposed to be Kähler. We fix symplectic forms ω_1 and ω_2 and we consider the group of symplectomorphisms. A *symplectomorphism* of symplectic manifolds (M, ω_1) and (N, ω_2) is a (smooth) diffeomorphism $\Psi : M \to N$ such that $\omega_1 = \Psi^*\omega_2$. The group of symplectomorphisms will be denoted by $Symp(M, N)$ or $Symp(M) = Symp(M, M)$. Finally, a mapping $\varphi : (M, J^g) \to (N, J^h)$ is *holomorphic* if $J^h \circ \varphi_* = \varphi_* \circ J^g$ and *anti-holomorphic* if $J^h \circ \varphi_* = -\varphi_* \circ J^g$.

1.1.2 Harmonic Mappings

First we recall the notion of a harmonic mapping $\varphi : (M, g) \to (N, h)$. For the moment we suppose M to be compact. The *energy density* of a smooth mapping $\varphi \in C^\infty(M, N)$ is defined by

$$\begin{aligned} e(\varphi)(x) = \frac{1}{2} Tr_g(\varphi^* h)(x) &= \frac{1}{2} \sum_{i=1}^{2n} (\varphi^* h)_x(v_i, v_i) \\ &= \frac{1}{2} \sum_{i=1}^{2n} h_{\varphi(x)}(\varphi_* v_i, \varphi_* v_i). \end{aligned} \tag{1}$$

Here $\{v_i\}_{i=1}^{2n}$ is an arbitrary basis of $T_x M$, orthonormal with respect to the metric g. The energy density is independent of this choice. The integral of the energy density

$$E(\varphi) := \int_M e(\varphi) dg$$

is called the *energy* (or action integral) of φ.
The mapping φ is called *harmonic* if φ is a critical point of the energy E on $C^\infty(M, N)$: for any smooth variation φ_t, $-\varepsilon < t < \varepsilon$, we should have

$$\frac{d}{dt} |_{t=0} E(\varphi_t) = 0.$$

More precisely, we call a family of mappings φ_t a smooth variation of φ if the mapping $\Phi : (-\varepsilon, \varepsilon) \times M \to N$ defined by

$$\Phi(t, x) := \varphi_t(x), \qquad -\varepsilon < t < \varepsilon$$

satisfies

$$\begin{cases} \Phi(0, x) = \varphi(x) & \forall x \in M \\ \Phi \in C^\infty((-\varepsilon, \varepsilon) \times M, N). \end{cases}$$

Any smooth variation φ_t defines via

$$V(x) := \frac{d}{dt} |_{t=0} \varphi_t(x)$$

a mapping from M into the tangent bundle TN such that

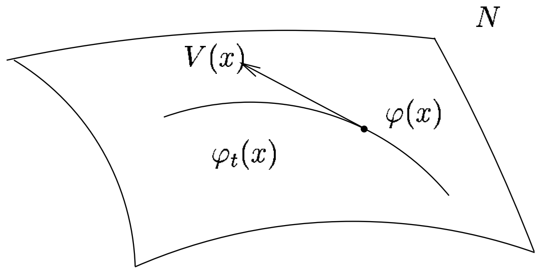
$$V(x) \in T_{\varphi(x)} N \qquad x \in M.$$

The vector field $V : M \to TN$ is called the *variation vector field* along φ. Conversely, given such a V satisfying the above property, we define a smooth variation with

$$\varphi_t(x) := \exp_{\varphi(x)}(tV(x))$$

with the property

$$\frac{d}{dt}\mid_{t=0}\varphi_t(x) = V(x).$$



We call a variation compactly supported if its variation vector field V is compactly supported in M. Since later on we will always have diffeomorphic mappings, we may as well work with the transported vector field

$$\begin{aligned} \tilde{V}: N &\rightarrow TN \\ y &\mapsto V(\varphi^{-1}(y)) \end{aligned}$$

which is more convenient for our purposes.

Noncompact Manifolds. If the manifold M is not compact we localize the problem: choose a open subset $M' \subset M$ with compact closure and compactly supported variation vector fields with support in M'. The mapping φ is harmonic if $\varphi\mid_{M'}$ is a critical point of the energy for all such variations and all M'.

1.1.3 Hamiltonian Isotopies

Consider a smooth isotopy $(-\varepsilon,\varepsilon)\times N \to N : (t,q)\mapsto \psi_t(q)$ such that $\psi_t \in Symp(N)$ are symplectomorphisms and $\psi_0 = id$. Such a family is called a *symplectic isotopy* of N. It is generated by a unique family of time-dependent vector fields $X_t : N \to TN$ such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

Since the mappings ψ_t are assumed to be symplectomorphic for all t, the vector fields X_t are locally Hamiltonian [7, p.88]: there exists a smooth family of Hamiltonian functions H_t, defined on a simply-connected subdomain U of N, satisfying

$$\omega(X_t,\cdot) = dH_t \qquad \text{on } U.$$

If H_t exists globally, ψ_t is called a *Hamiltonian isotopy.* On a simply-connected manifold N every symplectic isotopy is Hamiltonian.
Now let $\varphi : M \to N$ denote a symplectomorphism and consider the variation vector field $V : M \to TN$ along φ and its generated variation $\varphi_t := \exp_{\varphi(x)}(tV(x))$.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ & \searrow^{\varphi_t} & \downarrow{\scriptstyle \varphi_t \circ \varphi^{-1}} \\ & & N \end{array}$$

The flow $\varphi_t \circ \varphi^{-1} : N \to N$ is generated by the time-independent vector field

$$\tilde{V} : N \to TN, \qquad \tilde{V}(y) := V(\varphi^{-1}(y))$$

and also satisfies $\varphi_0 \circ \varphi^{-1} = id_N$. By the above, $\tilde{V} = X_t$ for all t and we see that locally the variation vector field $\tilde{V}$ is the Hamilton vector field associated to a *time-independent* Hamiltonian H:

$$\tilde{V}(y) = X_H(y), \qquad X_H := J^h \nabla^h H.$$

Here $\nabla^h H$ denotes the gradient of H taken with respect to the Riemannian metric $h(X,Y) = \omega_2(X, J^h Y)$, cf. [15, p.15].

We are primarily interested in deriving differential equations for symplectic maps that are critical points for the energy under symplectic variations. For this purpose it is enough to consider only local variations which are then given by a compactly supported Hamiltonian which will be extended by 0 to all of N.

Definition 1.2 *A smooth, compactly supported variation $\varphi_t : (-\varepsilon, \varepsilon) \times M \to N$ is called* symplectic *if its variation vector field $\tilde{V} : N \to TN$ is (globally) Hamiltonian.*

1.1.4 The Euler-Lagrange Equation

Harmonic mappings are solutions of a variational problem and satisfy therefore a system of partial differential equations, the Euler-Lagrange equation of the problem (this is the *Dirichlet principle*). Our aim here is to derive a system of differential equations for the restricted variational problem formulated for symplectic mappings.

Let φ_t, $-\varepsilon < t < \varepsilon$, be a smooth compactly supported variation of φ with variation vector field V of support $\mathrm{supp}(V) = M' \subset M$. The *first variation formula for the energy* reads as follows [41, p. 131]

$$\frac{d}{dt}\Big|_{t=0} E(\varphi_t) = -\int_{M'} h(V(x), \tau(\varphi)(x))dg. \qquad (2)$$

Here the vector field $\tau : M \to TN$, $\tau : x \mapsto \tau(\varphi)(x)$ is the so-called *tension field* of φ. It is the trace of the second fundamental form of φ; for this and further formulas see [9, 41]. The first variation formula shows that the vanishing $\tau(\varphi) = 0$ of the tension field is equivalent to φ being harmonic. However, for a symplectic variation, the vector field V must be Hamiltonian and thus the condition $\tau(\varphi) = 0$ will be relaxed. So, in this case, we use $V(x) = X_H(\varphi(x))$ to get

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E(\varphi_t) &= -\int_{M'} h(V(x), \tau(\varphi)(x))dg \qquad \mathrm{supp}(V) \subset M' \\
&= -\int_{M'} h(X_H(\varphi(x)), \tau(\varphi)(x))dg \qquad X_H \text{ Hamiltonian} \\
&= -\int_{M'} h(J^h \nabla^h H(\varphi(x)), \tau(\varphi)(x))dg \\
&= +\int_{M'} h(\nabla^h H(\varphi(x)), J^h \tau(\varphi)(x))dg
\end{aligned}$$

since the Riemannian metric $h(X,Y) = \omega_2(X, J^h Y)$ is J^h-invariant. Using the abbreviations $\sigma(\varphi) := J^h \tau(\varphi)$ and $y := \varphi(x)$,

$$\begin{aligned}
&= +\int_{\varphi(M')} h(\nabla^h H(y), \sigma(\varphi)(\varphi^{-1}(y)))dh \qquad \det J_\varphi = 1, \text{ change of variables} \\
&= -\int_{\varphi(M')} H(y) \cdot \mathrm{div}^h \sigma(\varphi)(\varphi^{-1}(y))dh. \qquad (3)
\end{aligned}$$

In the last step we used the integration by parts formula $\int f \cdot \mathrm{div}^h(X)dh = -\int h(\nabla^h f, X)dh$. The vanishing of (3) for arbitrary H with arbitrary compact support is only possible if

$$\mathrm{div}^h \sigma(\varphi)(\varphi^{-1}(y)) = 0 \qquad \forall y \in N$$

or equivalently

$$\mathrm{div}^h \sigma(\varphi) = 0. \qquad (4)$$

Proposition 1.3 *A symplectomorphism $\varphi \in Symp(M,N)$ is symplectically-harmonic if and only if*

$$div^h \sigma(\varphi) = 0$$

everywhere on M.

The 3^{rd}-order system of partial differential equations (4) is in general nonlinear and non-elliptic and therefore difficult to solve. For existence results for related minimization problems see the section on the direct method of the calculus of variations.

Example. Consider the Euclidean space $\mathbb{R}^4$, equipped with the Euclidean metric. We denote the coordinates with (x_1, x_2, y_1, y_2) and the components of $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ with φ^i. Choose the standard symplectic form $\omega_0 = \sum_{i=1}^2 dx_i \wedge dy_i$. Then the components of the tension field are $\tau^i(\varphi) = \Delta\varphi^i$ and we get

$$\mathrm{div}\ \sigma(\varphi) = \mathrm{div}\ J\tau(\varphi) = \frac{\partial}{\partial y_1}\Delta\varphi^1 + \frac{\partial}{\partial y_2}\Delta\varphi^2 - \frac{\partial}{\partial x_1}\Delta\varphi^3 - \frac{\partial}{\partial x_2}\Delta\varphi^4.$$

Example. As an illustration, we give a very simple symplectic diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ which is symplectically-harmonic but not harmonic: the mapping

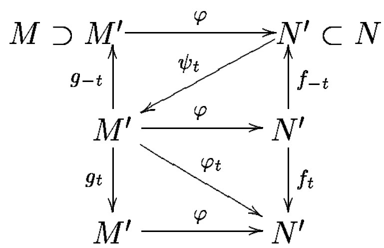
$$(x, y) \mapsto (x + y^2, x + y + y^2)$$

has all these properties. Its inverse $(r, s) \mapsto (r - (s - r)^2, s - r)$ is also an example.

Remark. Suppose a situation where we have a symplectically-harmonic mapping which is not harmonic (e.g. between two strictly pseudoconvex domains in $\mathbb{C}^n$ that are symplectomorphically but not biholomorphically equivalent). We could then try to apply the theory of *quasiharmonic fields* developed by Iwaniec and Sbordone ([18, 19], see also the last chapter) to the *div–curl couple* $\mathcal{F} = [\sigma, \nabla H]$. Here the vector field $\sigma = J^h\tau$ is divergence–free and the gradient $\nabla^h H$ of an arbitrary Hamiltonian function is curl–free. It would be interesting to calculate the distortion $\mathcal{K}$ of such couples and to decide when they are quasiharmonic fields. Once the calculations above have been generalized to include non–smooth mappings, a successful application of this theory would imply certain higher regularity results for symplectically-harmonic mappings.

1.1.5 Symmetry

In this section we show that the inverse of a symplectically-harmonic mapping is again symplectically harmonic. This is not surprising since the symplectomorphisms form a group under composition.
The mapping $\varphi : M \to N$ will always denote a symplectically-harmonic mapping. Let X_H be a Hamiltonian vector field with compact support $\mathrm{supp}(X_H) = N' \subset N$. The time-t mapping of its flow will be denoted by f_t in order to distinguish it from the Hamiltonian isotopy $\varphi_t = f_t \circ \varphi$. The symplectic diffeomorphism φ establishes a one-to-one correspondence between Hamiltonian isotopies on N and on M. The pull-back $\varphi^* X_H := (\varphi^{-1})_* X_H$ of X_H is again Hamiltonian since $\varphi^* X_H = X_{H \circ \varphi}$, cf. [7, p.18]. Therefore, the time-t maps g_t of the pull-back are symplectomorphisms and the isotopy $\psi_t := g_t \circ \varphi^{-1}$ is Hamiltonian. Finally, set $M' := \varphi^{-1}(N')$. By definition we have $f_t \circ \varphi = \varphi \circ g_t$ and the following diagram is commutative:



Lemma 1.4
$$\int_{M'} e(\varphi_t)(x)dg = \int_{N'} e(\varphi_t^{-1})(y)dh \qquad y := \varphi_t(x).$$

Proof.
Choose an orthonormal basis (X_i) of T_xM with respect to g. Using the Hermitian property of the metrics and $\varphi_t^*\omega_2 = \omega_1$ we deduce

$$\begin{aligned} h(\varphi_{t*}X_i, \varphi_{t*}X_i) &= \omega_2(\varphi_{t*}X_i, J^h\varphi_{t*}X_i) = \varphi_t^*\omega_2(X_i, \varphi_{t*}^{-1}J^h\varphi_{t*}X_i) \\ &= \omega_1(X_i, \varphi_{t*}^{-1}J^h\varphi_{t*}X_i) = g(X_i, (J^g)^{-1}\varphi_{t*}^{-1}J^h\varphi_{t*}X_i) \end{aligned}$$

and therefore (recall formula (1))

$$e(\varphi_t)(x) = \frac{1}{2}\sum_{i=1}^{2n} g(X_i, (J^g)^{-1}\varphi_{t*}^{-1}J^h\varphi_{t*}X_i) = \frac{1}{2}Tr(J^g)^{-1}\varphi_{t*}^{-1}J^h\varphi_{t*}.$$

Repeating this for an orthonormal basis (Y_i) of T_yN gives

$$e(\varphi_t^{-1})(y) = \frac{1}{2}Tr(J^h)^{-1}\varphi_{t*}J^g\varphi_{t*}^{-1}.$$

Equality follows now from $(J^h)^2 = -id$ and $(J^g)^2 = -id$:

$$\begin{aligned} Tr(J^g)^{-1}\varphi_{t*}^{-1}J^h\varphi_{t*} &= -Tr(J^g\varphi_{t*}^{-1})(J^h\varphi_{t*}) = -Tr(J^h\varphi_{t*})(J^g\varphi_{t*}^{-1}) \\ &= Tr(J^h)^{-1}\varphi_{t*}(J^g)\varphi_{t*}^{-1} \end{aligned}$$

and the lemma is proved.

Let E_1 and E_2 denote the energies of ψ_t and φ_t^{-1}:

$$\begin{aligned} E_1(t) &:= \int_{N'} e(\psi_t)(y)dh = \int_{N'} e(g_t \circ \varphi^{-1})(y)dh \\ E_2(t) &:= \int_{N'} e(\varphi_t^{-1})(y)dh = \int_{N'} e(\varphi^{-1} \circ f_{-t})(y)dh. \end{aligned}$$

From the commutativity of the above diagram we get $g_t \circ \varphi^{-1} = \varphi^{-1} \circ f_t$ which implies

$$E_1(t) = E_2(-t) \qquad -\varepsilon < t < \varepsilon. \tag{5}$$

Proposition 1.5 *A symplectomorphism φ is symplectically-harmonic if and only if φ^{-1} is symplectically-harmonic.*

The proposition follows from the lemma and (5):

$$\begin{aligned} \frac{d}{dt}|_{t=0}E(\varphi_t) &= \frac{d}{dt}|_{t=0}\int_{M'} e(\varphi_t)(x)dg = \frac{d}{dt}|_{t=0}E_2(t) \\ \frac{d}{dt}|_{t=0}E(\psi_t) &= \frac{d}{dt}|_{t=0}\int_{N'} e(\psi_t)(y)dh = \frac{d}{dt}|_{t=0}E_1(t). \end{aligned}$$

1.2 Symplectic Mappings of Sobolev Class

1.2.1 Definition and First Properties

We shall now define symplectic mappings of Sobolev class. For the sake of simplicity, we will do so only for mappings between domains Ω in Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The symplectic form will always be the standard symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. There is no real loss of generality in doing so since $(\mathbb{R}^{2n}, \omega_0)$ is the generic local model for all symplectic manifolds by the theorem of Darboux.
Recall that a (smooth) symplectic mapping (or *symplectomorphism*) is a diffeomorphism $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ such that $f^*\omega_0 = \omega_0$. In the coordinates chosen above, this condition takes the form

$$Df(x)^T J Df(x) = J \qquad \forall x \in \mathbb{R}^{2n}$$

where J is the skew-symmetric matrix defined by $J = \begin{pmatrix} & -Id_n \\ Id_n & \end{pmatrix}$.
From this we conclude that the Jacobian determinant satisfies $J_f^2 = 1$. It turns out that in fact $J_f = 1$. This is easily seen using differential forms: the $2n$-form

$$\Omega := \omega_0 \wedge \omega_0 \wedge \cdots \wedge \omega_0 \quad n \text{ times}$$

is a volume form on $\mathbb{R}^{2n}$

$$\Omega = c \cdot dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$$

where $c \neq 0$ is a constant. From $f^*\omega_0 = \omega_0$ it follows $f^*\Omega = \Omega$ and using the fact $f^*\Omega = J_f \cdot \Omega$ we conclude $J_f = 1$.

We propose the following definition for non-smooth symplectic mappings:

Definition 1.6 *Let $\Omega \subset \mathbb{R}^{2n}$, $n \geq 1$, be a domain. A mapping $f \in W^{1,2n}_{loc}(\Omega, \mathbb{R}^{2n})$ is symplectic (with respect to the standard symplectic form ω_0) if*

$$Df(x)^T J Df(x) = J \qquad \textit{for almost every } x \in \Omega. \tag{6}$$

Some remarks on this definition are in order. The derivative matrix Df is the formal derivative of f; its elements are the weak (distributional) partial derivatives of the coordinates of f which exist as L^{2n}_{loc}-functions. The definition *does not* suppose that f is differentiable in the ordinary sense. Note also that mappings in $W^{1,2n}_{loc}(\Omega, \mathbb{R}^{2n})$ need not have a continuous representative. These regularity properties will be consequences of the symplecticity condition (6). The important point here is that the Jacobian determinant J_f of f is almost everywhere equal to 1. In particular, the Jacobian does not change sign.

The symplectic mappings are special mappings of finite dilatation in the sense of the following definition [20]

Definition 1.7 *Let $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, $1 \leq p \leq \infty$, and let $|Df(x)|$ designate the supremum norm of the matrix $Df(x)$. The mapping f has finite dilatation (or is a mapping of finite dilatation) if a measurable function K_f exists such that*

i) The bound $1 \leq K_f < \infty$ holds for almost every $x \in \Omega$,

ii) For almost every $x \in \Omega$,

$$|Df(x)|^n \leq K_f(x) \cdot J_f(x). \tag{7}$$

Such a function K_f is then called a dilatation function *of f.*

This class contains the maps with *bounded dilatation* (when $||K_f||_\infty \leq K$ for some constant K), nowadays more widely known as *quasiregular* mappings. Any mapping $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ almost everywhere is of finite dilatation; in particular, this applies to symplectic mappings. Despite the seemingly weak conditions, mappings of finite dilatation retain some important analytical properties of quasiconformal mappings. The following properties will be of fundamental importance for further calculations:

Theorem 1.8 *Let $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ be a mapping of finite dilatation. Then*

1. *f has a continuous representative (also called f) satisfying*

$$|f(x) - f(y)| \leq C(n)||Df||_{L^n} \left(\log \frac{2}{|x-y|}\right)^{-1/n} \qquad x, y \in K \subset \Omega \text{ compact,}$$

2. *f is differentiable pointwise almost everywhere in the ordinary sense,*

3. *f satisfies Lusin's conditions (N) and (N^{-1}), i.e. the image of a set with zero Lebesgue n-measure is a set of zero measure and also the preimage of a zero-set is a zero-set.*

Remarks: Property 1 is a consequence of the fact that these mappings are monotone. The original proof goes back to 1977 and is due to Gol'dshtein and Vodop'yanov ([42], see also [37, p. 339]). A more recent proof of assertion 1 can be found in [28].
The estimate of the modulus of continuity given in 1 shows that in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ a bounded family of maps is equicontinuous. The proof of 2 is the same as for mappings with bounded dilatation; equicontinuity is enough for the proof [37, I.1.2].
The Lusin condition (N) implies in particular that the image of a measurable set is measurable and it plays a major role in many change-of-variables formulas. Even a continuous mapping of Sobolev class $W^{1,n}$ need not satisfy the Lusin condition as an example of Malý

and Martio [26] shows. They construct a mapping of class $W^{1,n}$ which maps a line segment onto the unit cube. The (N^{-1})-property follows from $J_f(x) > 0$ almost everywhere ([42, 2.3, 2.4]). For a detailed account of the Lusin (N)-condition see [26, 30].
Recent advances in the theory of quasiregular mappings might suggest that the integrability assumption $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ could be relaxed [17]. An example by Ponomarev [34] indicates that the exponent n is indeed "critical": For all $1 \leq p < n$, he constructed a one-to-one mapping $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. which does not have property (N).

Example. It is a non-trivial task to give symplectic examples with prescribed regularity. The most natural way of constructing symplectic mappings is to obtain them as time–1 mappings of a Hamiltonian flow. A step towards a theory of non-smooth Hamiltonian systems has been done e.g. by Künzle [24], but a precise theory linking the properties of the flow and the vector field seems to be missing. In the cited article it is shown that non-smooth systems can violate fundamental properties valid for smooth systems. For example, a symplectic mapping derived from a non-convex Hamiltonian H (see the article for the precise definition) need not preserve the level sets of H.
At least, we have the following, admittedly somewhat trivial, example:
Take $\lambda_1, \lambda_2 \in W^{1,p}(\mathbb{R})$ and consider the shear-map $\mathbb{R}^4 \to \mathbb{R}^4$, $(x_1, y_1, x_2, y_2) \mapsto (x_1', y_1', x_2', y_2')$ given by

$$\begin{aligned} x_1' &= x_1 + \lambda_1(y_1) y_1 \\ y_1' &= y_1 \\ x_2' &= x_2 + \lambda_2(y_2) y_2 \\ y_2' &= y_2. \end{aligned}$$

This mapping is symplectic since $dx_i' \wedge dy_i' = (dx_i + \lambda_i(y_i) dy_i + y_i \frac{\partial \lambda_i}{\partial y_i} dy_i) \wedge dy_i = dx_i \wedge dy_i$, and belongs to $W^{1,p}_{loc}(\mathbb{R}^4, \mathbb{R}^4)$.
More examples result from the thesis of Gratza [12]. He showed that smooth symplectic mappings can be approximated uniformly with piecewise linear symplectic mappings.

1.2.2 Direct Method of the Calculus of Variations

In the first section we derived the Euler-Lagrange equation for symplectically-harmonic mappings, but we couldn't provide criteria for solvability of this problem. Here we will show that some minimization problems have symplectic solutions. This will be achieved using the Direct Method of the Calculus of Variations and noting that the class of symplectic mappings is closed under weak convergence.

Definition 1.9 *Let X be a Banach space with norm $||\cdot||$. A functional $I : X \to \mathbb{R}$ is said to be sequentially weakly lower semicontinuous (swlsc) if*

$$\liminf_{n \to \infty} I(u_n) \geq I(u)$$

whenever $u_n \rightharpoonup u$ in X. The functional I is coercive on a subset *$\mathcal{A}$ of X if $||u_n|| \to \infty$ for $u_n \in \mathcal{A}$ implies $I(u_n) \to \infty$.*

Theorem 1.10 *Let $\mathcal{A}$ be a weakly closed subset of a reflexive Banach space X and let $I : X \to \mathbb{R}$ be sequentially weakly lower semicontinuous and coercive on $\mathcal{A}$. Then I attains a minimum an $\mathcal{A}$.*

Proof: Choose a minimizing sequence u_k in $\mathcal{A}$, i.e. a sequence of $u_n \in \mathcal{A}$ such that $I(u_k) \to \inf_{u\in\mathcal{A}} I(u)$. The coercivity condition forces u_k to be bounded in the reflexive Banach space X. Therefore we can extract a subsequence $u_{k_j} \rightharpoonup \bar{u}$ converging weakly to some $\bar{u} \in \mathcal{A}$. The theorem follows as the functional I is swlsc:

$$I(\bar{u}) \leq \liminf_{j\to\infty} I(u_{k_j}) = \inf_{u\in\mathcal{A}} I(u).$$

□

The swlsc property is in general not easy to check; the discussion of necessary and sufficient conditions for it makes up a significant part in the literature in the calculus of variations. We will now place us in a more concrete setting and then choose a class of functionals that will be broad enough for our purposes.
For the rest of this section we consider functionals of the form

$$I(u) = \int_\Omega f(x, Du(x))dx \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ will always be a bounded domain with sufficiently smooth boundary, $u : \Omega \to \mathbb{R}^n$, $f : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}$ continuous. The reflexive Banach space X will typically be $W_0^{1,p}(\Omega, \mathbb{R}^n)$, $1 < p < \infty$.

Theorem 1.11 *[29] Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is continuous and $g(x, \cdot)$ convex for all x. Let $w_j \in L^1_{loc}(\Omega, \mathbb{R}^m)$ be a sequence which converges as a distribution to $w \in L^1_{loc}(\Omega, \mathbb{R}^m)$. Then*

$$\liminf_{j\to\infty} \int_\Omega g(x, w_j(x))dx \geq \int_\Omega g(x, w(x))dx.$$

This theorem, although not stated in the most general way, applies already to a large class of functionals as the next theorem shows.

Theorem 1.12 *[6, 4.2.6] Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $1 < p < \infty$. Let u_j converge weakly to $\bar{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$. If $p \geq k$ then any $k \times k$ – minor of Du_j converges in the sense of distributions to the corresponding minor of $D\bar{u}$.*

The assertion of the theorem might be surprising at first sight. The minors consist of products of weakly convergent sequences of functions and these products will a priori not converge anymore. But this reasoning is too crude. Weak continuity is a consequence of the cancellation effects due to the special structure of the minors; any minor can be expressed as divergence of a suitable vector field ([6, 4.2.7]). The very existence of nonlinear weakly continuous functions is a strictly higher-dimensional phenomenon.

We could then ask to identify all weakly continuous functions $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that $f(Du_j) \to f(Du)$ converges as distribution for every sequence $u_j \rightharpoonup u$ converging weakly in some suitable $W^{1,p}(\Omega, \mathbb{R}^n)$. This question has been solved by Reshetnyak [38, 39] and Ball [1, 2, 4]: a continuous f is sequentially weakly continuous if and only if it can be written as an affine combination of the minors (subdeterminants) of any size of Du. The Jacobian $f(Du) = J_u$ is such an example.
We see that we may take any minor for w_j in theorem 1.11. The most general function g depends on all minors of all sizes k (as long as $p \geq k$) and there are $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k}$ of them. A function f is called *polyconvex* if it can be written as a convex function of the minors. Theorem 1.11 and 1.12 show that $I(u) = \int_\Omega f(x, \nabla u)dx$ is swlsc if f is polyconvex.

Examples of lower semicontinuous functionals.

1. ([6, A.1]) Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ and let A be a $n \times n$–matrix. Recall that the singular values $v_1, \ldots, v_n$ of A are the eigenvalues of the positive semidefinite symmetric matrix $\sqrt{AA^T}$. If f satisfies $f(A) = f(QA)$ for all $Q \in SO(n)$ and $f(A) = f(QAQ^T)$ for all $Q \in O(n)$, then there exists $\Phi : (\mathbb{R}_+)^n \to \mathbb{R}$, symmetric in the variables (that is, invariant under permutations of the variables), such that $f(A) = \Phi(v_1, \ldots, v_n)$. Then f is convex if and only if Φ is convex and non-decreasing in each variable v_i. This result comes handy when dealing with symplectic mappings:
Recall the Cartan decomposition
$$A = U_1 D U_2$$
of a symplectic matrix A. Here $U_1, U_2 \in U(n) \cong O(2n) \cap Sp(n)$ are unitary and the diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})$ consists of the singular values $\lambda_i, \lambda_i^{-1} > 0$ of A.

2. Let $f \in W^{1,m}_{loc}(\Omega, \mathbb{R}^m)$ be a mapping of finite dilatation ($|A|$ is the sup-norm of a matrix A):
$$|Df(x)|^m \leq K_f(x) \cdot J_f(x) \qquad \text{almost every } x \in \Omega.$$
The dilatation function (more precisely, the *outer dilatation*)
$$K_f(x) := \begin{cases} |Df(x)|^m J_f(x)^{-1} & J_f(x) > 0 \\ 1 & otherwise \end{cases}$$
is polyconvex on matrices with positive Jacobian [11] and therefore the functionals
$$K_p(f) := \int_\Omega K_f^p(x)dx \qquad p \geq 1$$
are weakly lower semicontinous. Consider also the *linear dilatation* defined by
$$H_f(x) := \begin{cases} \frac{\max\{|Df(x)h| : h \in S^{n-1}\}}{\min\{|Df(x)h| : h \in S^{n-1}\}} & J_f(x) > 0 \\ 1 & otherwise \end{cases}.$$

In contrast to above, the linear dilatation is not lower semicontinous, see [16]. For symplectic mappings however, we can express it as $H_f(x) = \max_{i=1}^n \lambda_i^2 = |Df(x)|^2$, which is a symmetric, convex and non-decreasing function in the singular values $(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})$ and hence

$$H^p(f) := \int_\Omega H_f^p(x)dx \qquad p \geq 1$$

are weakly lower semicontinuous.

3. The p-energy of a mapping $f : (\mathbb{R}^n, \langle,\rangle) \to (\mathbb{R}^n, \langle,\rangle)$ with respect to the Euclidean metric $\langle,\rangle$,

$$\begin{aligned} E_p(f) &:= \int_\Omega e(f)(x)^{p/2}dx = \left(\frac{1}{2}\right)^{p/2} \int_\Omega (TrDf(x)^T Df(x))^{p/2}dx \\ &= \left(\frac{1}{2}\right)^{p/2} \int_\Omega \left(\sum_{i,j} |Df(x)|_{ij}^2\right)^{p/2} dx = \frac{1}{2}||Df||_{L^p}^p, \end{aligned}$$

is clearly lower semicontinuous. Lower semicontinuity for the energy $E = E_2$ is also known for mappings $f \in L^2(M,N)$ between Riemannian manifolds [21]. In this general situation subtle problems arise already when defining the notion of Sobolev space.

1.2.3 Existence of Symplectic Extremal Mappings

We show that some minimization problems can be solved within the class of symplectic mappings. First we fix this class of admissible mappings for the variation. We suppose that $u_0 \in \mathcal{A}$ is some given symplectic mapping. Typically, u_0 is a smooth symplectomorphism from Ω onto its image Ω'. We assume that Ω is a Sobolev extension domain (later we need the Poincaré–inequality). Let

$$\mathcal{A} := \{u \in W^{1,2n}(\Omega, \mathbb{R}^{2n}) : u \text{ symplectic}, u - u_0 \in W_0^{1,2n}(\Omega, \mathbb{R}^{2n})\}.$$

Definition 1.13 *We call $u \in \mathcal{A}$ a symplectic extremal mapping of the functional $I : W^{1,2n}(\Omega, \mathbb{R}^{2n}) \to \mathbb{R}$ for the boundary values u_0 if $I(u) \leq I(w)$ for all $w \in \mathcal{A}$.*

We can now state the existence results:

Theorem 1.14 *Let $p \geq 1$, $q \geq n$, $r \geq 2n$. Then the functionals K_p, H_q and E_r attain a minimum on $\mathcal{A}$, i.e. there exists an extremal for the boundary values u_0.*

The proof is virtually the same as in the abstract setting. First, we check that the admissible class $\mathcal{A}$ is closed under weak convergence.

Proposition 1.15 *Let* (u_j) *be a sequence of symplectic mappings converging weakly in* $W^{1,2n}_{loc}(\Omega, \mathbb{R}^{2n})$. *Then the limit mapping* $\bar{u}$ *is symplectic.*

The proof of the proposition makes essential use of the weak continuity of minors of the matrix $Du_j(x)$.

Proof: The weakly convergent sequence Du_j is bounded in $L^{2n}(\Omega, \mathbb{R}^{2n\times 2n})$. The adjoint $A^\sharp$ of a matrix A is the transposed matrix of the $(2n-1)$-minors of A with signs and has the property $A^\sharp A = \det A \cdot Id$. The adjoint of a symplectic matrix A is given by

$$A^\sharp = A^{-1} = J^T A^T J.$$

Applying this to Du_j we deduce the boundedness of $D^\sharp u_j$ in $L^{2n}(\Omega, \mathbb{R}^{2n\times 2n})$. Thus, we may extract a weakly convergent subsequence of u_j (still denoted by u_j) such that

$$D^\sharp u_j \rightharpoonup M \quad \text{ in } L^{2n}(\Omega, \mathbb{R}^{2n\times 2n}).$$

This sequence converges also in the sense of distributions. Theorem 1.12 asserts

$$D^\sharp u_j \to D^\sharp \bar{u}.$$

The limits $D^\sharp \bar{u}$ and M must be the same by the uniqueness of distributional limits. Now we have to show that $M = D^\sharp \bar{u}$ is symplectic. First, a similar reasoning shows $J_{\bar{u}} = 1$ almost everywhere. The symplecticity condition $A^T J A = J$ can be equivalently stated in the form

$$A^\sharp = J^T \cdot A^T \cdot J$$

provided $\det A = 1$. From

$$\begin{aligned} D^\sharp u_j &= J^T \cdot D^T u_j \cdot J \\ D^\sharp u_j &\rightharpoonup D^\sharp \bar{u} \quad \text{ in } L^{2n}(\Omega, \mathbb{R}^{2n\times 2n}) \end{aligned}$$

and the convergence

$$J^T \cdot D^T u_j \cdot J \rightharpoonup J^T \cdot D^T \bar{u} \cdot J$$

we conclude

$$D^\sharp \bar{u}(x) = J^T \cdot D^T \bar{u}(x) \cdot J \qquad a.e.\ x \in \Omega$$

and the proof is finished. □

Proof of 1.14: Let I denote one of the functionals of the proposition.
Choose a minimizing sequence (u_j) of I in $\mathcal{A}$:

$$u_j \in \mathcal{A}\ , \qquad \lim_{j\to\infty} I(u_j) = \inf\{I(u) : u \in \mathcal{A}\}.$$

Note that $\mathcal{A}$ contains u_0 by assumption and $I(u) < \infty$ for $u \in \mathcal{A}$. It remains to check that I is coercive. Recall the Poincaré–inequality

Lemma 1.16 *Let Ω be a bounded Sobolev extension domain. Then for $u \in W_0^{1,p}(\Omega)$, $1 \le p < \infty$,*

$$\left(\int_\Omega |u|^p dx\right)^{1/p} \le C(p,\Omega)\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}.$$

It states that on the space $W_0^{1,p}(\Omega)$ the Sobolev norm $||u||_{W^{1,p}} = (||u||_{L^p}^p + ||\nabla u||_{L^p}^p)^{1/p}$ is equivalent to $||\nabla u||_{L^p}$.
By applying the Poincaré inequality to $u_j - u_0$ componentwise we see that all the functionals of the theorem are coercive. In fact, for $p = 1$, $q = n$, $r = 2n$ the functionals grow like the L^{2n}-norm of Du_j. This lower bound is also valid for higher exponents since the power function is convex (use Jensen's inequality).
Now the proof proceeds as in the abstract setting. A subsequence of u_j converges weakly to a limit mapping $\bar{u}$. This limit satisfies $\bar{u} - u_0 \in W_0^{1,2n}(\Omega, \mathbb{R}^{2n})$ and is symplectic by proposition 1.15. Thus $\bar{u} \in \mathcal{A}$ and the proof is finished. □

1.2.4 Extremal Symplectic Mappings for Noncompact Manifolds

So far we have seen that some minimization problems can be solved abstractly within the class of symplectic mappings. The problem really gets interesting and much more demanding when we consider variational problems involving general Riemannian manifolds, especially noncompact ones. Since their volume is infinite, it is in general not possible to minimize some functional directly on the whole manifold. This applies at least for many functionals we are interested in (i.e. those that are coercive with respect to the Sobolev norm of the manifold). An obvious idea is to solve local variational problems and to define the global mapping by a second limit process. We present such an approach here and describe a problem that could not be solved.
Consider the following model situation:
Let $\Phi : (\overline{\Omega}, \omega_0) \to (\overline{\Omega'}, \omega_0)$ be a given smooth symplectomorphism between subdomains of $(\mathbb{R}^{2n}, \omega_0)$. Suppose these domains carry a Riemannian metric g of infinite volume and we want to minimize a functional

$$I^\Omega(u) := \int_\Omega F(x, Du(x))dg(x)$$

which is coercive on the Sobolev space corresponding to g. Our aim is to construct an extremal mapping by exhaustion and a normal family argument: Let Ω_k be an exhaustion by domains $\Omega_k \subset \Omega_{k+1} \subset \Omega$, $\cup\Omega_k = \Omega$. Suppose that F is such that the local minimization problems

$$I^{\Omega_k}(u) \leq I^{\Omega_k}(v), \qquad v \text{ symplectic}, v - \Phi|_{\Omega_k} \in W_0^{1,2n}(\Omega_k, \mathbb{R}^{2n})$$

are solvable in $W^{1,2n}(\Omega_k, \mathbb{R}^{2n})$ and give a family of (continuous) symplectic mappings $\{u_k\}$. We extend u_k to $W^{1,2n}(\Omega, \mathbb{R}^{2n})$.

If we could show that $\{u_k\}$ is locally pointwise bounded and locally equicontinuous, then the theorem of Arzelà– Ascoli would give us the desired continuous (and monotone, since this is preserved under local uniform convergence) limit mapping.

Both local boundedness and equicontinuity follow if $\{u_k\}$ is locally bounded in Sobolev norm, say in $W^{1,2n}(D, \mathbb{R}^{2n})$ where $D \subset \Omega$ is fixed (theorem 1.8).

We have been unable to prove this. Although $I_k(u_k)$ is minimal amongst all admissible mappings with boundary values given by $\Phi|_{\Omega_k}$, it is not clear whether $||u_k||_{W^{1,2n}(D,\mathbb{R}^{2n})}$ stays bounded. It could be that $I^D(u_k) \to \infty$ despite of $I^{\Omega_k}(u_k) \leq I^{\Omega_k}(\Phi)$ since $I^{\Omega_k}(\Phi)$ grows like the weighted Sobolev norm. The boundary values $u_k|_{\partial D}$ are different from $\Phi|_{\partial D}$ and we cannot argue that $I^D(u_k) \leq I^D(\Phi)$. It is hard to imagine that a monotone u_k should behave very wildly on D and, because of $u_k - \Phi|_{\Omega_k} \in W_0^{1,2n}(\Omega_k, \mathbb{R}^{2n})$, be very tame near $\partial\Omega_k$. One should have a good estimate on the oscillation on D keeping account of the distance from D to $\partial\Omega_k$ so that the growth of $||u_k||_{W^{1,2n}(D,\mathbb{R}^{2n})}$ will be compensated by the growing distance. The Harnack inequality for monotone functions f in [28] doesn't solve the problem because it also depends on the local norm of $\log f$.

2 Global Invertibility of Symplectic Mappings

2.1 The Mapping Degree

Let $\Omega \subset \mathbb{R}^{2n}$ be a nonempty bounded open set and let $u : \overline{\Omega} \to \mathbb{R}^n$ be a continuous mapping. Consider a nonempty bounded domain $G \subset\subset \Omega$. A point $p \in \mathbb{R}^n$ is called (u, G)-admissible if $yp \notin u(\partial G)$. The Brouwer mapping degree (also called local degree or topological index) of u with respect to G assigns to every (u, G)-admissible point p an integer $d(u, G, p)$. The mapping degree satisfies a list of axioms (solution property, naturality, excision, additivity, homotopy invariance and normalization) which in turn determine d uniquely. The degree is constant on the connected components V of $\mathbb{R}^n \backslash u(\partial G)$. We denote the common value with $d(u, G, V)$. Furthermore, it depends only on the boundary values of u. For mappings $u \in C^1(G)$ we can calculate the degree with the formula

$$d(u, G, V) = \int_G \rho(u(x)) J_u(x) dx \tag{9}$$

where ρ is an arbitrary non-negative real-valued continuous function with compact support in V and satisfying the normalization $\int_V \rho(y)dy = 1$. For this formula and a detailed exposition of degree theory see [40]. We show now that the formula (9) remains valid for continuous mappings $u \in W^{1,n}(\Omega, \mathbb{R}^n)$. Both sides of (9) are still defined for such mappings. The continuous mapping u can be approximated by a sequence of smooth mappings u_ε such that ([32], p.11):

i) $u_\varepsilon \to u$ locally uniformly

ii) $u_\varepsilon \to u$ in $L^n(\Omega)$

iii) For all $G \subset\subset \Omega$: $Du_\varepsilon \to Du$ in $L^n(\Omega)$.

Such an u_ε can be defined by mollifying with a Sobolev averaging kernel:

$$u_\varepsilon(x) := \frac{1}{\varepsilon^n} \int \varphi(\frac{x-y}{\varepsilon}) u(y) dy, \qquad \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0, \quad \int_\Omega \varphi(x) dx = 1.$$

Now take an (u, G)–admissible point p, i.e. $p \notin u(\partial G)$. For ε small enough, p is still admissible for u_ε by locally uniform convergence. The degree is invariant under homotopies and it follows $d(u, G, p) = d(u_\varepsilon, G, p)$ for ε small enough. Formula (9) is valid for u_ε and it remains to check that the right-hand side converges but this follows from property iii) above (note $G \subset\subset \Omega$). Thus we have checked the validity of (9) for continuous $u \in W^{1,n}(\Omega)$.

It's possible find a better approximation [37, p. 87]:

Lemma 2.1 *Let G be the closure of a bounded domain in $\mathbb{R}^n$, and let $u : G \to \mathbb{R}^n$ be continuous. Then for every (u, G)–admissible point p there exist a sequence of smooth mappings $u_\varepsilon : G \to \mathbb{R}^n$ such that u_ε converges to u uniformly on G and each of the mappings is* regular *with respect to p, i.e. p is (u_ε, G)–admissible, the set $u^{-1}(p)$ is finite and the Jacobian is nonzero at the points of $u^{-1}(p)$.*

Definition 2.2 *Let $f : G \to \mathbb{R}$ be a function and $A \subset G$. The* multiplicity $N(u|A, y, f)$ *of a mapping $u : G \to \mathbb{R}^n$ at a point $y \in \mathbb{R}^n$ is defined by*

$$N(u|A, y, f) := \sum_{x \in u^{-1}(y) \cap A} f(x).$$

In particular, for the constant $f = 1$, we denote by $N(u|A, y) = N(u|A, y, 1)$ the cardinality of $u^{-1}(y) \cap A$. The function $N(u|A, y)$ may be infinite.

The assertions of the following theorem are the same as in [3], but the assumptions are different: our mappings are only in the class $W^{1,n}(\Omega, \mathbb{R}^n)$ and are therefore not Hölder-continuous. Furthermore, we do not require the boundary ∂G of G to be Lipschitz; we only need that it has zero Lebesgue n-measure.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain and $u \in W^{1,n}(\Omega, \mathbb{R}^n)$. Suppose $J_u(x) > 0$ almost everywhere in Ω. Let $u_0 : \Omega \to \mathbb{R}^n$ be continuous and injective in Ω. Let $G \subset\subset \Omega$ be a relatively compact domain with $|\partial G| = 0$ and $u|\partial G = u_0|\partial G$. Then*

1) $u(\overline{G}) = u_0(\overline{G})$

2) u maps measurable sets in $\overline{G}$ to measurable sets in $u_0(\overline{G})$ and the change of variables formula

$$\int_A f(u(x)) J_u(x) dx = \int_{u(A)} f(v) dv$$

holds for any measurable $A \subset \overline{G}$ and any measurable $f : \mathbb{R}^n \to \mathbb{R}$ if one of the integrals exist.

3) u is one–to–one almost everywhere: the set

$$S = \{v \in u_0(\overline{G}) : u^{-1}(v) \textit{ contains more than one element}\}$$

has measure zero.

4) if $v \in u_0(G)$ then $u^{-1}(v)$ is a continuum (closed and connected) in G. If $v \in \partial u_0(G)$ then each connected component of $u^{-1}(v)$ intersects ∂G.

Remarks

1. The assumption $J_u(x) > 0$ a.e. implies that u is a mapping of finite dilatation; see section 1.2.1 for a discussion. In particular, u is monotone and has a continuous representative. We will always assume that the continuous representative has been chosen. The unbounded component of $\mathbb{R}^n \backslash u(\partial G)$ does not contain points of $u(G)$ since u is monotone [37, p. 176].

2. The mapping u is continuous up to (and beyond) the boundary of the subdomain G by the assumption of the theorem. There is no need to construct an extension to a bigger domain and therefore G need not be a Sobolev extension domain. This condition will however be needed for the proof of the main theorem.

3. Let u be symplectic. Then $J_u(x) = 1$ almost everywhere and u is therefore measure-preserving according to 2). This does not trivially imply $|S| = 0$.

4. Assertion 3) follows from 4) with the theorem

Theorem 2.4 *[30] Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous mapping. Then $f^{-1}(y)$ is totally disconnected for almost all $y \in \mathbb{R}$.*

Proof of theorem 2.3: [3]
1) The invariance of domain theorem implies that u_0 is a homeomorphism of G onto the open set $u_0(G)$. Therefore $u_0(\overline{G}) = \overline{u_0(G)}$ and $\partial u_0(G) = u_0(\partial G)$. Furthermore

$$\begin{aligned} d(u_0, G, u_0(G)) &= \pm 1 \\ d(u_0, G, p) &= 0 \quad p \in \mathbb{R}^n \backslash u_0(\overline{G}) \end{aligned}$$

For the first statement see [[40], p.98] or [[35], IV.4.6]. The second is just the solution property of the degree. Since $J_u(x) > 0$ a.e. formula (9) implies $d(u, G, p) \geq 0$ (u is sense–preserving) and since $u|\partial G = u_0|\partial G$ it follows that

$$\begin{aligned} d(u, G, u_0(G)) &= 1 \\ d(u, G, p) &= 0 \quad p \in \mathbb{R}^n \backslash u_0(\overline{G}) \end{aligned} \tag{10}$$

Therefore if $p \in u_0(G)$ then $u^{-1}(p)$ is not empty. Hence $u(\overline{G}) \supset u_0(\overline{G})$. Let $p \notin u_0(\overline{G})$ and suppose for contradiction that $u(x) = p$ for some $x \in G$. Apply (9) with V the component of $\mathbb{R}^n \backslash u_0(\overline{G})$ containing p and with ρ strictly positive in a neighbourhood U of p. The continuity of u implies that a small ball around x is mapped into U. Since $J_u(x) > 0$ a.e. the right–hand side of (9) is positive. This contradiction proves 1).

4) Let $v \in u_0(G)$ and suppose that the closed set $u^{-1}(v)$ is not connected. Then there exist nonempty subsets M_1, M_2, E_1, E_2 of G with E_1, E_2 open, such that $M_1 \cap M_2$ and $E_1 \cap E_2$ are empty, $u^{-1}(v) = M_1 \cup M_2$, $M_1 \subset E_1$ and $M_2 \subset E_2$. Since $v \notin u(\partial E_1) \cup u(\partial E_2)$ and since $J_u(x) > 0$ a.e., the degrees $d(u, E_i, v)$, $i = 1, 2$, given by (9) are defined and positive for p in a neighbourhood U of v. Therefore $U \subset u(E_1) \cap u(E_2)$, a contradiction to 3). The argument for $v \in \partial u_0(G)$ is similar: suppose that a connected component M of $u^{-1}(v)$ does not intersect ∂G. Choose open sets E_1 and E_2 of Ω with $M \subset E_1$, $E_1 \cap E_2 = \emptyset$ and E_2 containing the (unique) point of $u^{-1}(v)$ on ∂G. Again $d(u, E_i, p)$, $i = 1, 2$, are positive for p in a neighbourhood U of v and the contradiction $U \subset u(E_1) \cap u(E_2)$ results.

2) and 3) Since u has the (N)–property it maps measurable sets $A \subset \bar{G}$ to measurable sets $u(A)$. There are several theorems about change of variables available, we use the following which can be applied in our situation:

Theorem 2.5 *[37, p. 99] Let Ω be an open set in $\mathbb{R}^n$, and $u : \Omega \to \mathbb{R}^n$ a continuous mapping. Assume that u has property (N) and is differentiable a.e. in Ω with locally integrable Jacobian. Then for every nonnegative measurable function $f : \Omega \to \mathbb{R}$ the function $y \mapsto N(u|\Omega, y, f)$ is measurable on $\mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} N(u|\Omega, y, f)dy = \int_\Omega f(x)|J_u(x)|dx \tag{11}$$

Further, if $G \subset \Omega$ is the closure of a bounded domain whose boundary has measure zero, then, for every nonnegative measurable function $f : \mathbb{R}^n \to \mathbb{R}$ such that the function $y \mapsto N(u|G, y)f(y)$ is integrable, the function

$$x \mapsto f(u(x))J_u(x)$$

is integrable over G and

$$\int_G f(u(x))|J_u(x)|dx = \int_{\mathbb{R}^n} f(y)N(u|G, y)dy \tag{12}$$

$$\int_G f(u(x))J_u(x)dx = \int_{\mathbb{R}^n} f(y)d(u, G, y)dy \tag{13}$$

In particular we take $f = 1$. Note that we have $u(\partial G) = \partial u_0(G)$ and $J_u(x) > 0$ a.e. We deduce that

$$\int_{u_0(G)} N(u|\overline{G}, v)dv = \int_{u^{-1}(u_0(G))} J_u(x)dx. \tag{14}$$

By formulas (9) and (10) we have

$$1 = \int_G \rho(u(x))J_u(x)dx \tag{15}$$

for any continuous, nonnegative ρ with supp$\rho \subset\subset u_0(G)$ and $\int_{\mathbb{R}^n} \rho(y)dy = 1$.
Now take $\psi_r \in C(\mathbb{R})$ with supp$\psi \subset\subset u_0(G)$, $0 \le \psi_r \le 1$, $\psi_r(p) = 1$ if $p \in u_0(G)$ and $dist(p, \partial u_0(G)) \ge \frac{1}{r}$. Let $\rho_r := \frac{1}{\int \psi_r(p)dp}\psi_r$ and apply (15) to it, passing to the limit using Lebesgue dominated convergence:

$$\begin{aligned} \int \psi_r(p)dp &= \int_G \psi_r(u(x))J_u(x)dx \\ \lim_{r\to\infty} \int \psi_r(p)dp &= \lim_{r\to\infty} \int_{u^{-1}(u_0(G))} \psi_r(u(x))J_u(x)dx \\ \int \lim_{r\to\infty} \psi_r(p)dp &= \int_{u^{-1}(u_0(G))} \lim_{r\to\infty} \psi_r(u(x))J_u(x)dx \\ |u_0(G)| &= \int_{u^{-1}(u_0(G))} J_u(x)dx. \end{aligned} \tag{16}$$

Combine (14), (16) and use $N(u|\overline{G}, v) \geq 1$ for $v \in u_0(G)$ and we see $N(u|\overline{G}, v) = 1$ a.e. in $u_0(G)$. Note $|\partial u_0(G)| = |u_0(\partial G)| = |u(\partial G)| = 0$ since $|\partial G| = 0$ and u has property (N). This proves 3) and assertion 2) follows from (12), applied to $f \cdot \chi_{u(A)}$.

2.2 Global Invertibility

One of the prominent features of the theory of (smooth) symplectic mappings is the symmetry resulting from the fact that symplectic diffeomorphisms form a group. Philosophically speaking, the inverse of a symplectic mapping should have the same (analytical, geometrical) properties as the mapping does. We have already seen one instance of this principle when we proved that the inverse of a symplectically-harmonic mapping is also harmonic. Our main result in this section is another example. Based on theorem 2.3 above, we show that symplectic mappings are homeomorphic.

Theorem 2.6 *Let the hypotheses of theorem 2.3 hold and let $u \in W^{1,2n}(\Omega, \mathbb{R}^{2n})$ be symplectic according to definition 1.6. Suppose $u_0(G)$ is a Sobolev extension domain. Then $u|G$ is a homeomorphism of G onto $u_0(G)$ and the inverse mapping is again symplectic and belongs to $W^{1,2n}(u(G), \mathbb{R}^{2n})$.*

Remark: The topological condition $u|\partial G = u_0|\partial G$ may not be omitted. An example [3] shows that then even local invertibility fails: the (symplectic!) mapping $u : \mathbb{D} \to \mathbb{D}$ of the unit disc in $\mathbb{R}^2$ given in polar coordinates as $u : (r, \varphi) \mapsto (\frac{1}{\sqrt{2}}r, 2\varphi)$ satisfies $u \in W^{1,\infty}(\mathbb{D})$ and $J_u(x) = 1$ if $x \neq 0$ but is not locally invertible at the origin.
This theorem is an adapted version of theorem 2 in [3]. Our result differs from that in some respects: First, we do not require a higher integrability $u \in W^{1,q}, q > 2n$. The original argument involving the cone condition will be replaced by an application of the estimate of the modulus of continuity for mappings of finite dilatation. The condition $\int_G |D^{-1}u(x)|^q J_u(x)dx < \infty$ which is necessary in general is automatically satisfied by symplectic mappings. Furthermore, the domain $u_0(G)$ need not be Lipschitz. For more information on Sobolev extension domains, see [13]. In the proof, we will follow closely the ideas and the organization of [3]. While we omit some calculations, we (re)produce a complete reasoning.

Proof: The inverse mapping $x(\cdot) \in W^{1,2n}(u_0(G), \mathbb{R}^{2n})$ will be constructed as a limit of a sequence of smooth mappings that may be seen as an "averaged inverse". If $x(\cdot)$ is indeed the inverse of u, then such a sequence is given by mollifying with a Sobolev averaging kernel $\rho_\varepsilon \in C_0^\infty(B_\varepsilon(0))$, $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}^{2n}} \rho_\varepsilon(v)dv = 1$

$$x_\varepsilon(v) = \int_{u_0(G)} \rho_\varepsilon(v - u)x(u)du$$

or

$$x_\varepsilon(v) = \int_G \rho_\varepsilon(v - u(y))yJ_u(y)dy. \tag{17}$$

Given only u, we *define* $x_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by (17) and prove that x_ε converges to the inverse of u. The calculations in [3, p. 321–322] show

$$\int_G \left|\frac{\partial x_\varepsilon^j(v)}{\partial v_i}\right|^{2n} dv \;\leq\; \int_G |(D^\# u(y))_{ij}|^{2n} (J_u(y))^{1-2n} dy \tag{18}$$

for ε sufficiently small. Here x_ε^j are the components of $x_\varepsilon = (x_\varepsilon^1, \ldots, x_\varepsilon^{2n})$ and $D^\#$ is the adjoint matrix of Du, i.e. the transpose of the matrix of the cofactors of Du. Surprisingly, this estimate is independent of ε. We apply $D^\# u = Du^{-1} = J^T Du^T J$ and $J_u = 1$

$$\sum_{i,j} \int_G \left|\frac{\partial x_\varepsilon^j(v)}{\partial v_i}\right|^{2n} dv = \sum_{i,j} \int_G |(Du(y))_{ij}|^{2n} dy \leq ||Du||^{2n}_{L^{2n}(\Omega)}. \tag{19}$$

Thus (x_ε) is bounded in $W^{1,2n}(u_0(G), \mathbb{R}^{2n})$ for sufficiently small ε and we can extract a subsequence (again denoted by (x_ε)) converging weakly in $W^{1,2n}(u_0(G), \mathbb{R}^{2n})$ to a mapping $x(\cdot)$.

The next step is to show that $x(\cdot)$ is a right inverse of u:

$$u(x(v)) = v \qquad \text{for all } y \in u_0(G). \tag{20}$$

Let S be the set where u is not injective. By theorem 2.3 we know $|S| = 0$. So for any $v \in u_0(G)\backslash S$ there is exactly one $x \in G$ with $u(x) = v$. Recall the definition of x_ε, equation (15) and $J_u = 1$:

$$x_\varepsilon(v) - x = \int_{u_0(G)} \rho_\varepsilon(u(x) - y) dy - \int_G x \rho_\varepsilon(u(y)) J_u(y) dy = \int_G \rho_\varepsilon(u(x) - u(y))(y - x) dy \tag{21}$$

for small $\varepsilon \leq \varepsilon_1$. Let $\eta > 0$. The uniqueness of x and the continuity of u imply the existence of a $\delta > 0$ such that $|y - x| < \eta$ whenever $|u(x) - u(y)| \leq \delta$. Otherwise we could construct a sequence converging to some point different from x but the image of this sequence would converge to v. If $\varepsilon \leq \min(\delta, \varepsilon_1)$ then

$$|x_\varepsilon(v) - x| \leq \eta \cdot \int_G \rho_\varepsilon(u(x) - u(y)) dy = \eta \cdot d(u, G, v) = \eta$$

and we conclude $x(v) = x$ and $u(x(v)) = v$ for all $v \in u_0(G)\backslash S$. This equality still holds for $v \in S$ by density, provided $x(\cdot)$ is continuous on $u_0(G)$. We show now that $x(\cdot)$ is symplectic and therefore has a continuous representative (in the following $x(\cdot)$ will always denote the continuous representative). Note that $x(\cdot)|_{u_0(G)\backslash S}$ is the inverse of u on the set $u^{-1}(u_0(G)\backslash S)$ which has full measure because of the (N^{-1})–property. Since the symplectic mapping u is differentiable in the ordinary sense a.e on this set, the inverse mapping $x(\cdot)$ is also differentiable almost everywhere and its derivative matrix is also symplectic almost everywhere on $u_0(G)$.

The mapping u is monotone; because of this we know that the unbounded component of $\mathbb{R}^{2n}\backslash u(\partial G) = \mathbb{R}^{2n}\backslash u_0(\partial G) = \mathbb{R}^{2n}\backslash \partial u_0(G)$ does not contain any points of $u(G)$. But it could happen that some point $x \in G$ is mapped to the boundary $\partial u_0(G)$. We will show that this is not possible in our situation. From our considerations so far, we can conclude that $B := x(u_0(G))$ is an open subset of G of full measure and $x : u_0(G) \to B$ is a monotone homeomorphism: Let $y \in B$ and take a sequence $y_r \in B$ with $y_r \to y$ and $u(y_r) \notin S$. By (20) and using that $u(y_r)$ has an unique inverse image under u (namely y_r) we get

$$x(u(y_r)) = y_r.$$

Using the continuity of $x(\cdot)$ and u we may take limits and obtain

$$x(u(y)) = y \qquad \text{for all } y \in B.$$

Thus u is a homeomorphism of B onto $u_0(G)$. This works also for points $y \in G$ once we know that the points are mapped to $u_0(G)$ and not to the boundary $\partial u_0(G)$. Therefore, the last step of the proof is to show $u : G \to u_0(G)$. We do this by contradiction.
The idea is the following: Assume $u(x_0) = y_0 \in \partial u_0(G), x \in G$. By theorem 2.3 the connected component of $u^{-1}\{y_0\}$ containing x_0 intersects the boundary of G. So it is a stretched-out set. On the other hand, a small ball around y_0 cannot get too large since we have a an estimate for the oscillation of the coordinate functions of $x(\cdot)$.
We begin the proof by choosing a preimage $x_0 \in u^{-1}\{y_0\} \cap G$ of $y_0 \in \partial u_0(G)$. Let $d = dist(x_0, \partial G) > 0$. The mapping $x(\cdot) \in W^{1,2n}(u_0(G), \mathbb{R}^{2n})$ can be extended to $\tilde{x}(\cdot) \in W^{1,2n}(u_0(G), \mathbb{R}^{2n})$ since $u_0(G)$ is supposed to be a Sobolev extension domain. The extended mapping is not monotone anymore. But it has a *2n–quasi–continuous* representative $\bar{x}(\cdot) \in W^{1,2n}(u_0(G), \mathbb{R}^{2n})$; i.e. $\tilde{x}$ can be redefined on a set of measure zero to a mapping $\bar{x}$ such that $\bar{x}$ is continuous outside a set of arbitrary small Sobolev $2n$–capacity [14, p. 87]. Let $y_0 = u(x_0)$ and $R > 0$ such that $\overline{B(y_0, R)} \subset u_0(\Omega)$. A consequence of q–quasi–continuity for $q > 2n - 1$ is that $\bar{x}$ is continuous on spheres $S(y_0, r)$ for radii $r \in (0, R)\backslash E(R)$ where the linear measure if $E(R)$ is zero, cf. ([M], p. 396). Further, collect all radii with $B \cap \bar{x}(S(y_0, r) \cap u_0(G)) = \emptyset$ into the set $F(R) \subset (0, R)$. Since B has full measure and x is measure–preserving in $u_0(G)$, the linear measure of $F(R)$ is zero, too. Pick a sequence of "good" radii $r_n \to 0$, i.e. $r_n \in (0, R)\backslash(E(R) \cup F(R))$. We apply the Sobolev inequality on spheres, also known as Gehring oscillation lemma [10]

$$\left(\operatorname{osc}_{S(y_0,r)} x_i\right)^p \le C_1(n,p) r^{p-(2n-1)} \int_{S(y_0,r)} |\nabla x_i|^p dS$$

to the coordinate functions x_i of $\bar{x}$ for the exponent $p = 2n$. Dividing both sides by r and integrating from 0 to R we get

$$\begin{aligned}\int_0^R \frac{(\operatorname{osc}_{S(y_0,r)} x_i)^{2n}}{r} dr &\le C_1 \cdot \int_0^R \left(\int_{S(y_0,r)} |\nabla x_i|^{2n} dS\right) dr \\ &= C_1 \cdot \int_{B_R(y_0)} |\nabla x_i|^{2n} dy < \infty.\end{aligned}$$

We insert our sequence of good radii r_n and note that there is a subsequence (still denoted by r_n) such that

$$\lim_{n\to\infty} \operatorname{osc}_{S(y_0,r_n)} x_i = 0$$

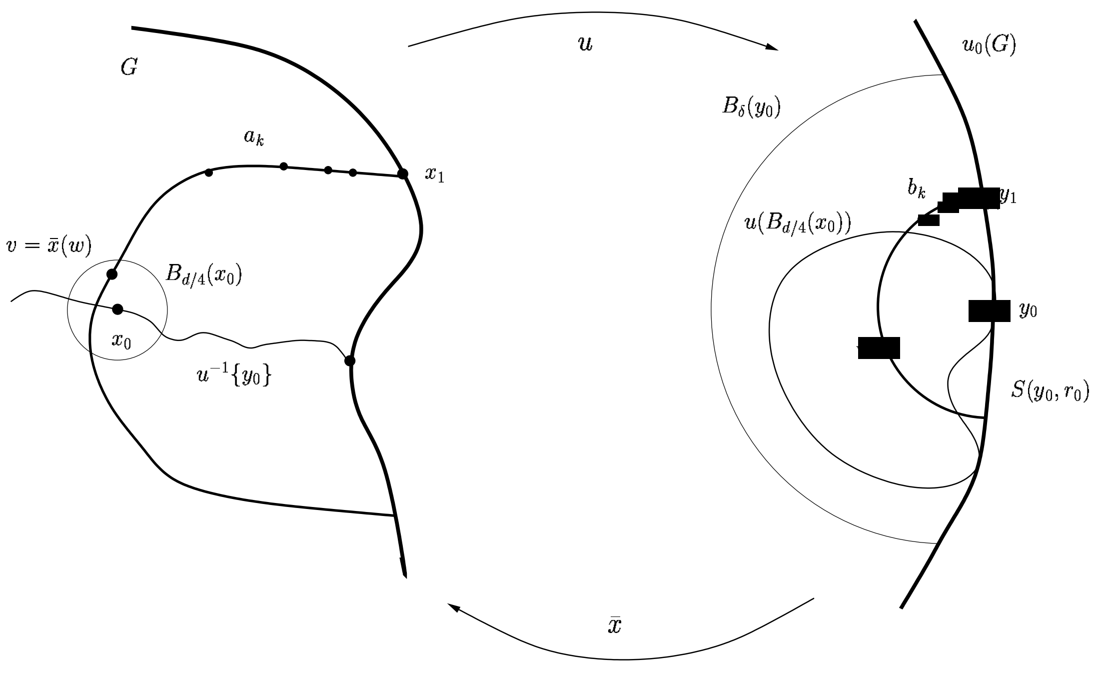
for some fixed $i = 1, \ldots, 2n$. Namely, if this would be false, the oscillation must be bounded away from 0, e.g. $\inf_r \operatorname{osc}_{S(y_0,r)} x_i = \alpha > 0$. Then the contradiction

$$\infty = \int_0^R \frac{\alpha^{2n}}{r} dr \le \int_0^R \frac{(\operatorname{osc}_{S(y_0,r)} x_i)^{2n}}{r} dr < \infty$$

results. By iteratively extracting subsequences we can find a sequence $r_n \to 0$ of good radii such that

$$\lim_{n\to\infty} \operatorname{osc}_{S(y_0,r_n)} x_i = 0$$

for $i = 1, \ldots, 2n$ simultaneously. Now take a ball $B_\delta(y_0)$ around the point $y_0 \in \partial u_0(G)$ and choose a neighbourhood U of x_0 such that $U \subset B_{d/4}(x_0)$ and $u(U) \subset B_\delta(y_0)$. Let $N \in \mathbb{N}$ be such that $r_n \le \delta$ for all $n \ge N$ and $\operatorname{osc}_{S(y_0,r_n)} \bar{x}_i < d/8$ for all i. For some $n \ge N$ large enough, the spherical cap $S(y_0, r_n) \cap u_0(G)$ intersects $u(B_{d/4}(x_0))$ in a point w. Otherwise the image of this ball would be disconnected which is impossible by the continuity of u or it would lie completely in the boundary which contradicts measure-preservation. Denote by r_0 the radius of this sphere with nonempty intersection. We know that $\bar{x}$ is continuous on this sphere.



To derive a contradiction look at the inverse image of the spherical cap $S(y_0, r_0) \cap \overline{u_0(G)}$ under $\bar{x}$. Let y_1 be some point of the boundary of the cap, i.e. $y_1 \in S(y_0, r_0) \cap \partial u_0(G)$. Choose a sequence of points $a_k \in \bar{x}(S(y_0, r_0) \cap u_0(G))$ converging to the point

$$x_1 := u^{-1}\{y_1\} \cap \partial G \in \bar{x}(S(y_0, r_0) \cap \partial u_0(G))$$

on the boundary of G. This point is unique since u is injective on the boundary. Recall that u is continuous up to the boundary (since, by assumption, u is continuous on an even bigger domain Ω) and thus the sequence $b_k := u(a_k)$ converges to y_1. Further, the mapping $\bar{x}$ is continuous on the *whole sphere* $S(y_0, r_0)$ and therefore the sequence $c_k := \bar{x}(a_k)$ converges to $\bar{x}(y_1)$.
But recall that on $u_0(G)$ we know that $\bar{x}|_{u_0(G)} = x(\cdot)$ is the inverse of $u|_B$. This implies $c_k = a_k$ and $\bar{x}(y_1) = x_1$. Now consider the image $v := \bar{x}(w)$ of the special point w in $S(y_0, r_0) \cap u_0(G) \cap u(B_{d/4}(x_0))$. We have

$$\begin{aligned} |x_0 - x_1| &\geq dist(x_0, \partial G) = d \\ |x_0 - v| &< d/4, \end{aligned}$$

and therefore

$$|x_1 - v| \geq 3d/4.$$

But by the choice of $n \geq N$ we also have

$$\mathrm{osc}_{S(y_0,r_0)} \bar{x}_i < d/8$$

which implies that x_2 should be contained in a cube with center x_1 and side–length $d/4$, a contradiction. Therefore it is impossible that a point in the interior of G is mapped onto the boundary, i.e. $u : G \to u_0(G)$. □

Corollary 2.7 *If the homeomorphism $u \in W^{1,2n}(\Omega, \mathbb{R}^{2n})$ is symplectic then u^{-1} is also symplectic and $u^{-1} \in W^{1,2n}(u(\Omega), \Omega)$.*

Proof: This follows by exhausting Ω with a sequence of such G_n so that the theorem can be applied. Since the bound (19) is independent of G_n we get the assertion using Lebesgue dominated convergence. □

3 Symplectic Mappings and the Beltrami Equation

Although the theory of quasiregular mappings in higher dimensions appears to be an excellent generalization of the classical one-dimensional theory of holomorphic functions in many respects, there is a lack of existence results and constructive methods. The class of symplectic quasiconformal mappings differs significantly from the whole class of quasiconformal mappings inasmuch as they fulfill a Beltrami system of equations that is an exact counterpart of the Beltrami equation in $\mathbb{C}$. The reason for this is the fact the deviation from conformality expresses at the same time the deviation from being holomorphic. This makes it possible to apply the tools of the theory of holomorphic mappings in several complex variables. In particular, the above-mentioned Beltrami system — subject to suitable conditions — can be solved. We also address the problem of constructing symplectic mappings with this method.

3.1 The Beltrami Equation

Throughout this section we consider domains Ω_i, $i = 1, 2$, in $\mathbb{C}^n = \{z_1, \ldots, z_n\}$. We shall identify $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ equipped with the standard complex structure $J = J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$, $J_0^2 = -id$ which is given by $J_0 : \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$. To simplify notations we set $X_i := \frac{\partial}{\partial x_i}$ and $Y_i := \frac{\partial}{\partial y_i}$. The matrix representation of J_0 in the basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is given by the familiar matrix $J_0 = \begin{pmatrix} & -I \\ I & \end{pmatrix}$. On the space of 1–forms it is given by $J_0 : dx^i \mapsto -dy^i, dy^i \mapsto dx^i$. The complex structure is compatible with the standard symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$; that is, the symmetric, bilinear form $g : T\mathbb{R}^{2n} \times T\mathbb{R}^{2n} \to \mathbb{R}$ defined by $g(v, w) := \omega_i(Jv, w)$ is positive definite (for our choice of J and ω it is just the Euclidean scalar product). Usually we work with the complexified tangent $T_{\mathbb{C}}\Omega_i$ and cotangent spaces $T^*_{\mathbb{C}}\Omega_i$ and extend real objects $\mathbb{C}$-linearly (only the scalar product g is extended by $g(v, w) = \omega(Jv, \overline{w})$). The complexified tangent space $T_{\mathbb{C}}\Omega_i$ decomposes into a direct sum

$$T_{\mathbb{C}}\Omega_i = T^{1,0}\Omega_i \oplus T^{0,1}\Omega_i \qquad (22)$$

of the eigenspaces of J,

$$\begin{aligned} T^{1,0}\Omega_i &:= \{X - iJX : X \in T\Omega_i\} = \{Z \in T_{\mathbb{C}}\Omega_i : JZ = iZ\} \\ T^{0,1}\Omega_i &:= \{X + iJX : X \in T\Omega_i\} = \{Z \in T_{\mathbb{C}}\Omega_i : JZ = -iZ\} \end{aligned}$$

which we call the spaces of holomorphic and anti–holomorphic vectors. A complex basis of them is given by

$$\begin{aligned} Z_j &:= \frac{1}{2}(X_j - iY_j) \qquad j = 1, \ldots, n \\ \overline{Z}_j &:= \frac{1}{2}(X_j + iY_j). \end{aligned}$$

The complexified cotangent spaces decompose accordingly

$$T^*_{\mathbb{C}}\Omega_i = T_{1,0}\Omega_i \oplus T_{0,1}\Omega_i = \{dZ \in T^*_{\mathbb{C}}\Omega_i : JdZ = idZ\} \oplus \{dZ \in T^*_{\mathbb{C}}\Omega_i : JdZ = -idZ\}$$

with a complex basis given by

$$\begin{aligned} dz^j &:= dx^j + idy^j \qquad j = 1, \dots, n \\ d\bar{z}^j &:= dx^j - idy^j. \end{aligned}$$

Next we recall the definition of the complex dilatation μ as derived in [22, 23].
Let $f : \Omega_1 \to \Omega_2$ be a symplectic mapping. The image f_*Z_j of a holomorphic vector Z_j may be decomposed, according to (22), into $f_*Z_j = V_j + \overline{W}_j$ where $V_j, W_j \in T^{1,0}$. This defines a $\mathbb{C}$-antilinear mapping $\mu : T^{1,0} \to T^{1,0}$ mapping V_j to W_j. Using bases $(Z_1, \dots, Z_n)$ of $T^{1,0}\Omega_1$ and $(Z_1', \dots, Z_n')$ of $T^{1,0}\Omega_2$ we can write

$$\begin{aligned} V_j &= \sum_{k=1}^{n} p_{kj} Z_k' \qquad P = (p_{kj}) \\ W_j &= \sum_{k=1}^{n} q_{kj} Z_k' \qquad Q = (q_{kj}). \end{aligned}$$

The vectors $V_1, \dots, V_n$ constitute a basis of $T^{1,0}\Omega_2$; otherwise there would be a $Z \in T^{1,0}\Omega_1$ such that $f_*Z = 0 + \overline{W} \in T^{0,1}\Omega_2$ and the contradiction

$$\begin{aligned} 0 < g(W, W) &= \omega(JW, \overline{W}) = i \cdot \omega(W, \overline{W}) = i \cdot \omega(\overline{f_*Z}, f_*Z) \\ &= i \cdot \omega(\overline{Z}, Z) = -g(Z, Z) < 0 \end{aligned}$$

would result. Thus there is a $\mathbb{C}$–antilinear mapping $\mu : T^{1,0}\Omega_2 \to T^{1,0}\Omega_2$ such that

$$W_j = \sum_{s=1}^{n} \mu_{sj} V_s \qquad \mu = (\mu_{sj}) = P^{-1}Q. \tag{23}$$

The matrix $\mu = (\mu_{sj})$ is the *complex dilatation* of f. The matrices P and Q come from the complexification of the derivative matrix $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ of f_*; the complexification $S_{\mathbb{C}}$ of this real symplectic matrix is

$$S_{\mathbb{C}} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \quad P = \frac{S_1 + S_4}{2} + i \cdot \frac{-S_2 + S_3}{2}, \quad Q = \frac{S_1 - S_4}{2} + i \cdot \frac{S_2 + S_3}{2}.$$

The Cartan decomposition for $S_{\mathbb{C}}$ is

$$S_{\mathbb{C}} = \begin{pmatrix} U_1 & 0 \\ 0 & \overline{U}_1 \end{pmatrix} \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} U_2 & 0 \\ 0 & \overline{U}_2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} \cosh t_1 & & \\ & \ddots & \\ & & \cosh t_m \end{pmatrix}, \qquad B = \begin{pmatrix} \sinh t_1 & & \\ & \ddots & \\ & & \sinh t_m \end{pmatrix}$$

and $U_i \in U(n)$. This implies in particular that μ_f is symmetric:

$$\mu_f = P^{-1}Q = (U_1 A U_2)^{-1}(U_1 B \overline{U}_2) = U_2^{-1} A^{-1} B \overline{U}_2.$$

Using $S_{\mathbb{C}}^{-1} = \begin{pmatrix} P^* & -Q^T \\ -Q^* & P^T \end{pmatrix}$ we also get $\mu_{f^{-1}} = -U_1 A^{-1} B U_1^T$.
The equations for μ_f and $\mu_{f^{-1}}$ show that the sup-norms $||\mu_{f^{-1}}|| = ||\mu_f||$ and, more important, that the norm $||\mu_f|| = \max_i || \tanh t_i||$ is related to the linear dilatation

$$H_f(p) := \frac{\max\{|Df(p)h| : |h| = 1\}}{\min\{|Df(p)h| : |h| = 1\}} \qquad p \in \Omega_i$$

of the real mapping $f : \Omega_1 \subset \mathbb{R}^{2n} \to \Omega_2 \subset \mathbb{R}^{2n}$ at a point p. Indeed, the Cartan decomposition of the real symplectic matrix S is $S = U_1 \cdot \operatorname{diag}(e^{t_1}, \dots, e^{t_n}, e^{-t_1}, \dots, e^{-t_n}) U_2$. This implies

$$\frac{H_f - 1}{H_f + 1} = \max_i || \tanh t_i||_\infty = ||\mu_f||_\infty.$$

A homeomorphism with $||\mu_f||_\infty \leq \kappa < 1$ is $\frac{\kappa+1}{\kappa-1}$–quasiconformal.

The equation $f_* Z_j = V_j + \sum_s \overline{\mu_{sj} V_j}$ is a decoupled system of partial differential equations for the coordinate functions $f^i : \mathbb{C}^n \to \mathbb{C}$ of $f = (f^1, \dots, f^n)$: set $\bar{\partial}_j := \frac{\partial}{\partial \bar{z}_j}$ and $\partial_s := \frac{\partial}{\partial z_s}$. Plugging $V_s = \sum_{k=1}^n \partial_s f^k Z_k'$ and $W_j = \sum_{k=1}^n \bar{\partial}_j f^k Z_k'$ into the definition (23) yields

$$\bar{\partial}_j f^k = \sum_{s=1}^n \mu_{sj} \partial_s f^k \qquad k = 1, \dots, n. \tag{24}$$

Remarkably, each of the coordinate functions f^k satisfies this Beltrami equation independently. The solution of this equation will be achieved by solving the following associated equation on the space of complex 1–forms:

$$\bar{\partial}_j f^k d\bar{z}^j = \sum_{s=1}^n \mu_{sj} \partial_s f^k d\bar{z}^j \qquad k = 1, \dots, n.$$

In this context we think of μ as a tensor $\mu = \sum_{j,s} \mu_{sj} d\bar{z}^j \otimes \frac{\partial}{\partial z^s}$ sending the $(1,0)$–form $\partial f^k := \sum_s \frac{\partial f^k}{\partial z_s} dz^s$ to the $(0,1)$-form

$$\mu \cdot \partial f^k := \sum_{j,s=1}^n \mu_{sj} \frac{\partial f^k}{\partial z_s} d\bar{z}^j$$

and the Beltrami equation can be stated as

$$\bar{\partial} f^k = \mu \cdot \partial f^k. \tag{25}$$

It is worthwhile to express (24) in real notation: let h be one of the coordinate functions f^k. The complex derivatives of $h = u + iv$ are

$$\begin{aligned} h_{\bar{z}} &:= (\bar{\partial}_1 h, \ldots, \bar{\partial}_n h) = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial y_1}, \ldots\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x_1} + i\frac{\partial v}{\partial y_1}, \ldots\right) \\ h_z &:= (\partial_1 h, \ldots, \partial_n h) = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial y_1}, \ldots\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x_1} - i\frac{\partial v}{\partial y_1}, \ldots\right). \end{aligned}$$

Decomposing also $\mu = M_1 + iM_2$ into real and imaginary parts and expanding the real form of (24)

$$(Re\ h_{\bar{z}} + iIm\ h_{\bar{z}}) = (M_1 + iM_2)\,(Re\ h_z + iIm\ h_z)$$

we get a system of $2n$ real equations:

$$\begin{pmatrix} Re\ h_{\bar{z}} \\ Im\ h_{\bar{z}} \end{pmatrix} = \begin{pmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{pmatrix} \begin{pmatrix} Re\ h_z \\ Im\ h_z \end{pmatrix}. \tag{26}$$

3.2 Solution of the Beltrami Equation

The Beltrami equation (24) can be solved by techniques that are formally very similar to the ones used in the solution of the classical Beltrami equation in $\mathbb{C}$. Two features however are peculiar to the higher dimensional situation. The first is the presence of cohomological obstructions: a $\bar{\partial}$-problem $\bar{\partial} f = \alpha$ is solvable only if $\bar{\partial}\alpha = 0$. The second peculiarity is the Newlander-Nirenberg integrability condition that imposes necessary restrictions on μ. We shall proceed as follows: first we discuss shortly the $\bar{\partial}$-problem and the integrability condition. Then we report the method of Wang [43] where the ideas are clearly visible. Finally we mention the technically more involved and stronger result of Zhuravlëv [46, 47]. The key tool in these proofs are integral representation formulas. Wang uses the formula

$$\alpha = T_1(\bar{\partial}\alpha) - \bar{\partial} T(\alpha) \tag{27}$$

where α is a (sufficiently smooth) differential form of type $(0,1)$ and T_1, T are certain integral operators. Formally, if $\bar{\partial}\alpha = 0$ then $F := -T(\alpha)$ solves $\bar{\partial} F = \alpha$ since $\bar{\partial} F = -\bar{\partial} T(\alpha) = \alpha - T_1(\bar{\partial}\alpha) = \alpha$. Similarly, Zhuravlëv defines operators I, Π, S on the space $L_p^{0,1}$ of $(0,1)$-forms with L^p-coefficients that satisfy

$$\partial I\alpha = \Pi\alpha \qquad \bar{\partial} I\alpha = S\alpha.$$

The singular integral operators $\Pi : L_p^{0,1} \to L_p^{1,0}$, $S : L_p^{0,1} \to L_p^{0,1}$ are continuous. Again, if $\bar{\partial}\alpha = 0$, then $F := I\alpha$ solves $\bar{\partial} F = \alpha$.

Formula (27) is a particular case of the Bochner-Martinelli-Koppelmann integral formula; for general results of this type see the monograph [25] and [36].
The integrability condition

$$\bar{\partial}_j \mu_{sk} - \bar{\partial}_k \mu_{sj} = \sum_{r=1}^{n} (\mu_{rj} \partial_r \mu_{sk} - \mu_{rk} \partial_r \mu_{sj}) \qquad j, k, s = 1, \ldots, n \tag{28}$$

is a set of necessary conditions for the existence of $2n$ linearly independent solutions of (24), provided μ is sufficiently smooth [33]. With the notation from [43] that we adopted in equation (25), the integrability condition can be written as

$$\bar{\partial}\mu = \mu \partial \mu.$$

Yet another formulation is useful [45]: the complex dilatation μ, satisfying the integrability condition, defines a new complex structure with associated derivatives ${}_\mu\partial$, ${}_\mu\bar{\partial}$.
Denote

$$\mu^s := \sum_{j}^{n} \mu_{sj} d\bar{z}^j$$

and take a (p, q)-form $\alpha \in T_{p,q}\Omega_1$. Define ${}_\mu\bar{\partial} : T_{p,q}\Omega_1 \to T_{p,q+1}\Omega_1$ as

$${}_\mu\bar{\partial}\alpha := \bar{\partial}\alpha - \sum_{s}^{n} \mu^s \wedge \partial_s.$$

Then the integrability condition takes the form

$${}_\mu\bar{\partial}\mu^s = 0 \qquad s = 1, \ldots, n.$$

Applying the new derivative to a function f^k gives

$${}_\mu\bar{\partial} f^f = \sum_{j=1}^{n} \left(\bar{\partial}_j f^k d\bar{z}^j - \sum_{s=1}^{n} \mu_{sj} d\bar{z}^j \wedge \partial_s f^k \right)$$

and the Beltrami equation reads

$${}_\mu\bar{\partial} f^k = 0.$$

Thus, solutions of it are holomorphic functions with respect to the new complex structure induced by μ.
We give now a formal sketch of the solution of (24). Suppose μ is sufficiently smooth and small in magnitude. Consider the auxiliary fixed-point equations

$$F = -\partial T(\mu F) + dz^k, \qquad F = \sum_{i=1}^{n} \alpha_i dz^i, \quad k = 1, \ldots, n. \tag{29}$$

The operator $\partial T\mu$ is contracting in a suitable Banach space and hence each equation has a unique solution F_k. Starting from the Bochner-Martinelli-Koppelmann formula

$$\mu F_k = T_1(\bar{\partial}(\mu F_k)) - \bar{\partial} T(\mu F_k),$$

using the fact that F_k solve (29) and in an essential way the integrability condition it can be calculated [43] that

$$\bar{\partial}(\mu F_k) = \mu \partial T_1(\bar{\partial}(\mu F_k)).$$

The operator $\mu\partial T_1$ is again contractive and $\bar{\partial}(\mu F_k) = 0$ follows. Then the functions

$$f^k := -T(\mu F_k) + z^k$$

are solutions of the Beltrami equation. Indeed

$$\begin{aligned} \bar{\partial} f^k &= -\bar{\partial} T(\mu F_k) = \mu F_k - T_1(\bar{\partial}(\mu F_k)) = \mu F_k \\ \partial f^k &= -\partial T(\mu F_k) + dz^k = F_k. \end{aligned}$$

The solution method of [46] works even for non-smooth μ: let Λ_p denote the norm of the continuous operator $\Pi : L_p^{0,1} \to L_p^{1,0}$. Suppose $p > r > 1$ and $q = pr/(p-r)$.
The conditions on μ are

(a) $\mu \in L^p(\mathbb{C}^n)$;

(b) $||\mu||_\infty \Lambda_p < 1$ and $||\mu||_\infty \Lambda_r < 1/2\sqrt{n-1}$;

(c) $\bar{\partial}_j \mu_{sk} \in L^q(\mathbb{C}^n) \cap L^r(\mathbb{C}^n) \qquad (j,k,s = 1,\ldots,n)$;

(d) $\partial_j \mu_{sk} \in L^p(\mathbb{C}^n) \qquad (j,k,s = 1,\ldots,n)$;

(e) the derivatives of $\partial_j \mu_{sk}$ are in $L^q(\mathbb{C}^n) \cap L^r(\mathbb{C}^n) \qquad (j,k,s = 1,\ldots,n)$.

Theorem 3.1 *Let $p > r > 2n$. Suppose conditions (a)–(e) hold. Then there exists a unique homeomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ in $W^{1,p}_{loc}(\mathbb{C}^n)$ that solves (24) and satisfies*

$$f(0) = 0, \qquad f(z) = z + O(|z|)^{1-2n/p} \quad \text{as } z \to \infty.$$

The derivatives $\bar{\partial}_s f^k$ are bounded and $\partial_s f^k - \delta_{ks} \in L^p$. The second derivatives of f^k are in L^r.

The symbol δ_{ks} is the Kronecker delta function. Condition (e) on the second derivatives of μ is only needed to show that the Jacobian determinant J_f of f is a solution of a special differential equation which then implies $J_f > 0$. Moreover, the assumption $p > r > 2n$ is not necessary for the construction of the solution; it is used to show continuity and the growth estimate. In fact, the solutions enjoy a better regularity (section 3.4).

3.3 On Symplectic Solutions

The solution of the Beltrami system that was constructed in the last section is only one particular solution among all solutions. Indeed, suppose μ is such that the theorem asserts a homeomorphic solution. Then any solution f admits a Stoilow type factorization $f = g \circ h$ where h is a homeomorphic solution and $g = (g^1, \dots, g^{2n})$ is holomorphic. For this it is enough to note [46] that the functions $g^i = f^i \circ h^{-1}$ satisfy $\bar{\partial} g^i = 0$. In particular, if Φ is a (homeomorphic) symplectic solution, then $\Phi = g \circ h$ with g biholomorphic.
Of course, we would like to construct the symplectic mapping Φ itself from a given μ. For this we have to

1. construct a symplectic matrix $\begin{pmatrix} P & P\mu \\ \overline{P\mu} & \overline{P} \end{pmatrix}$ from μ,
2. modify the solution method such that the solution satisfies $\partial_j \Phi^i = P_{ij}$.

The linear algebra task 1) is not difficult.

Lemma 3.2 *Let $\mu \in M_n(\mathbb{C})$ be symmetric and suppose for the sup-norm $||\mu|| < 1$. Then there exists a symplectic matrix with complex dilatation μ.*

Proof: A real matrix S is symplectic if and only if its complexification $S_{\mathbb{C}} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}$ satisfies

$$\begin{array}{llll} (I) & PP^* - QQ^* = I & (III) & P^*P - Q^T\overline{Q} = I \\ (II) & -PQ^T + QP^T = 0 & (IV) & P^*Q - Q^T\overline{P} = 0. \end{array}$$

By definition $Q = P\mu$. Equation (II) is always satisfied because $\mu = \mu^T$. Equation (I) gives a candidate for P:

$$\begin{aligned} PP^* - P\mu\overline{\mu}P^* = P\,(I - \mu\overline{\mu})\,P^* &= I \\ (I - \mu\overline{\mu}) &= (P^*P)^{-1} \\ P^*P &= (I - \mu\overline{\mu})^{-1}. \end{aligned}$$

The operator $(I - \mu\overline{\mu})^{-1}$ is hermitian since $(\mu\overline{\mu})^* = \mu\overline{\mu}$. Therefore it is diagonalizable and its eigenvalues are real. Furthermore the operator is positive since $||\mu|| < 1$ implies that all eigenvalues of $I - \mu\overline{\mu}$ are positive. Now we define P as the unique square root of this positive operator,

$$P := \sqrt{(I - \mu\overline{\mu})^{-1}}.$$

The uniqueness of the square root and $(P^*)^2 = (P^2)^* = [(I - \mu\overline{\mu})^{-1}]^* = (I - \mu\overline{\mu})^{-1} = P^2$ imply $P^* = P$ or $P^T = \overline{P}$. Equation (I) follows. For (III) note $P^*P = P^2 = (I - \mu\overline{\mu})^{-1}$ and $Q^T\overline{Q} = \mu P^T \overline{P}\overline{\mu}$. Then (III) is equivalent to

$$\begin{aligned} Q^T\overline{Q} &= (I - \mu\overline{\mu})^{-1} - I \\ \Leftrightarrow (I - \mu\overline{\mu})\,\mu P^T\overline{P}\overline{\mu} &= I - (I - \mu\overline{\mu}) = \mu\overline{\mu}. \end{aligned}$$

Calculating using von Neumann series

$$\begin{aligned}
\mu P^T\overline{P} &= \mu\overline{P^2} = \mu\overline{(I-\mu\overline{\mu})^{-1}} = \mu\left(I + \overline{\mu}\mu + \overline{\mu}\mu\overline{\mu}\mu + \dots\right) \\
&= \left(I + \mu\overline{\mu} + \mu\overline{\mu}\mu\overline{\mu} + \dots\right)\mu \\
&= \left(I - \mu\overline{\mu}\right)^{-1}\mu
\end{aligned} \tag{30}$$

we see that (III) holds. For the last equation use $P^*Q = PQ = P^2\mu = (I - \mu\overline{\mu})^{-1}\mu$, $Q^T\overline{P} = Q^T P^T = (PQ)^T = (P^2\mu)^T = \mu\overline{P}^2$ and apply again (30). □

It is not clear that the Beltrami system is solvable with the particular P from the lemma; we cannot expect that the necessary integrability condition is satisfied. But the lemma shows at least that there are no algebraic obstructions. Generally, it is possible to get solutions with suitable prescribed derivatives. For example, we could replace the fixed-point equation

$$F_k = -\partial T(\mu F_k) + dz^k$$

by

$$P_k = -\partial T(\mu P_k) + \alpha^k$$

where α^k are exact $(1,0)$-forms. We can solve this for $\alpha^k = (I + \partial T\mu)P_k$. Let us assume that $\bar{\partial}\alpha^k = 0$; then $\bar{\partial}(\mu P_k) = 0$ follows again. Set $f^k = -T(\mu P_k) + \beta_k$ where β_k satisfies $\bar{\partial}\beta^k = 0$ and $\partial\beta_k = \alpha_k$. Then $f = (f^1, \dots, f^n)$ would indeed be the desired solution

$$\begin{aligned}
\overline{\partial} f^k &= -\bar{\partial}T(\mu P_k) - T_1(\bar{\partial}\mu P_k) = \mu P_k \\
\partial f^k &= -\partial T(\mu P_k) + \alpha_k = P_k.
\end{aligned}$$

The symmetry of μ is a consequence of the J-invariance of the symplectic form [22]. Vice versa, we have

Proposition 3.3 *If the diffeomorphism $f \in C^\infty(\Omega, \mathbb{R}^{2n})$ has a symmetric complex dilatation then there exists a J-invariant symplectic form α such that $f^*\alpha = \omega$.*

Proof: We define α by $\alpha = (f^{-1})^*\omega$. Take vector fields $V = X - iJX$ and $W = Y - iJY$ in $T^{1,0}$. The mapping f maps the vector fields $V - \overline{\mu V}$ and $W - \overline{\mu W}$ into $T^{1,0}$ [22]. Then

$$\begin{aligned}
f^*\alpha(V - \overline{\mu V}, W - \overline{\mu W}) &= \alpha(f_*(V - \overline{\mu V}), f_*(W - \overline{\mu W})) = \alpha(V, W) \\
&= \omega(V - \overline{\mu V}, W - \overline{\mu W}) \\
&= \omega(V, V) - \omega(\overline{\mu V}, W) - \omega(V, \overline{\mu W}) + \omega(\overline{\mu V}, \overline{\mu W}) \\
&= 0
\end{aligned}$$

where the symmetry was used in the last line. Thus,

$$\begin{aligned}
\alpha(V, W) &= 0 \qquad \forall V, W \in T^{1,0} \\
\alpha(\overline{V}, \overline{W}) &= 0
\end{aligned}$$

and J-invariance follows from

$$\alpha(V,W) = \alpha(X,Y) - \alpha(JX,JY) - i \cdot [\alpha(JX,Y) - \alpha(X,JY)] = 0.$$

Moreover $d\alpha = d(f^{-1})^*\omega = (f^{-1})^* d\omega = 0$ and the non-degeneracy $\alpha \wedge \cdots \wedge \alpha \neq 0$ follows from $f^*(\alpha \wedge \cdots \wedge \alpha) = \omega \wedge \cdots \wedge \omega \neq 0$. □

3.4 Higher Regularity

The solutions of the Beltrami equation that were constructed above are K-quasiregular for some K. Therefore they are "higher integrable": there are constants $q = q(K) < 2n$ and $p = p(K) > 2n$ such that any K-quasiregular solution $f \in W^{1,q}(\Omega, \mathbb{R}^{2n})$ actually belongs to $f \in W^{1,p}(\Omega, \mathbb{R}^{2n})$, cf. [17]. In fact the gain of regularity is much bigger: the unexpected possibility to treat the decoupled Beltrami system with the $\bar{\partial}$-techniques leads to higher regularity already for exponents q close to 2. To prove this, take a coordinate function $h = f^k : \mathbb{C}^n \to \mathbb{C}$, $h = u + iv$ of a $W^{1,q}_{loc}(\mathbb{C}^n, \mathbb{C}^n)$–solution of (24) and associate to it a pair of vector fields $\mathcal{F} = [B, E]$. We will show that $\mathcal{F}$ is a so-called *K-quasiharmonic field* or a *div-curl couple with bounded distortion.* These terms were coined by Iwaniec and Sbordone in [18, 19].
We define two real vector fields by

$$\begin{aligned} E &:= \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n} \right) \\ B &:= J\nabla v = \left(\frac{\partial v}{\partial y_1}, \ldots, \frac{\partial v}{\partial y_n}, -\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right). \end{aligned}$$

Clearly,

$$\begin{aligned} \text{curl } E &= \text{curl } \nabla u = 0 \\ \text{div } B &= \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i \partial y_i} - \frac{\partial^2 v}{\partial y_i \partial x_i} = 0. \end{aligned}$$

The complex structure J can also be interpretated as a matrix representation of the Hodge $*$-operator which transforms curl-free vector fields to div-free vector fields (or closed differential forms to co-closed ones). A pair $\mathcal{F} = [B, E]$ with div $B = 0$, curl $E = 0$ is called a *div-curl couple.* The $\mathcal{F}^+$-component and the $\mathcal{F}^-$-component of $\mathcal{F}$ are defined as

$$\mathcal{F}^- := \frac{1}{2}(E - B) \qquad \mathcal{F}^+ := \frac{1}{2}(B + E).$$

We relate the components of the coordinate function $h = u + iv$ solving (24) to its complex derivatives: from

$$
\begin{aligned}
Re\ h_{\bar z} &= \frac{1}{2}\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) - \frac{1}{2}\left(\frac{\partial v}{\partial y_1},\dots,\frac{\partial v}{\partial y_n}\right) &&= \frac{1}{2}\,(E-B)_{i=1}^{n} \\
Im\ h_{\bar z} &= \frac{1}{2}\left(\frac{\partial u}{\partial y_1},\dots,\frac{\partial u}{\partial y_n}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x_1},\dots,\frac{\partial v}{\partial x_n}\right) &&= \frac{1}{2}\,(E-B)_{i=n+1}^{2n} \\
Re\ h_{z} &= \frac{1}{2}\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial y_1},\dots,\frac{\partial v}{\partial y_n}\right) &&= \frac{1}{2}\,(E+B)_{i=1}^{n} \\
Im\ h_{z} &= -\frac{1}{2}\left(\frac{\partial u}{\partial y_1},\dots,\frac{\partial u}{\partial y_n}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x_1},\dots,\frac{\partial v}{\partial x_n}\right) &&= -\frac{1}{2}\,(E+B)_{i=n+1}^{2n}
\end{aligned}
$$

we get

$$
\mathcal{F}^- = \begin{pmatrix} Re\ h_{\bar z} \\ Im\ h_{\bar z} \end{pmatrix}, \qquad \mathcal{F}^+ = \begin{pmatrix} Re\ h_z \\ -Im\ h_z \end{pmatrix}.
$$

Equation (26) translates to a Beltrami equation for $\mathcal{F}$,

$$
\mathcal{F}^-(x) = \tilde\mu(x)\cdot\mathcal{F}^+(x), \qquad \tilde\mu := \begin{pmatrix} M_1 & +M_2 \\ M_2 & -M_1 \end{pmatrix}. \tag{31}
$$

Let $||\mu||_\infty = \kappa = \frac{K-1}{K+1} < 1$. Since $||\mu||_\infty = ||\tilde\mu||_\infty$ we get $K = \frac{||\tilde\mu||_\infty+1}{||\tilde\mu||_\infty-1}$. Thus, in the terminology of [19], the pair $\mathcal{F}$ is a K-quasiharmonic field. The gain of regularity for the vector fields of such a pair is remarkable:

Theorem 3.4 *[19, thm. 2]*
Let a Hölder conjugate pair $q < 2 < p$ be given by

$$
q = \frac{14K-12}{7K-5} \qquad \textit{and} \qquad p = \frac{14K-12}{7K-7}, \qquad K \geq 1.
$$

Then every K-quasiharmonic pair of vector fields in $L^q_{loc}(\Omega, \mathbb{R}^{2n}\times\mathbb{R}^{2n})$ actually belongs to $L^p_{loc}(\Omega, \mathbb{R}^{2n}\times\mathbb{R}^{2n})$.

Note that $q \to 1$ and $p \to \infty$ as $K \to 1$. Quasiharmonic fields with $K = 1$ correspond to a single vector field $E = B$ which is harmonic since curl E = div $E = 0$. Therefore it is smooth according to Weyl's lemma. This theorem in conjunction with theorem 3.1 shows that there are effective tools to construct solutions of the Beltrami equation. Unfortunately it is not clear if it is always possible to find a *symplectic* solution.

4 Symplectic Mappings and Exterior Algebra

4.1 Exterior Powers of Vectors and Mappings

The l-th exterior power $\wedge^l(V)$ of a vector space V is characterized by the following universal property: any alternating multilinear mapping of l variables $f : V \times \cdots \times V \to W$ to some vector space W factors through $\wedge^l(V)$; there is a linear mapping $\varphi : \wedge^l(V) \to W$ such that

$$\begin{array}{ccc} V \times \cdots \times V & \xrightarrow{\ f\ } & W \\ {\scriptstyle i}\downarrow & \nearrow_{\varphi} & \\ \wedge^l(V) & & \end{array}$$

is commutative. The mapping i is the natural inclusion $(v_1, \ldots, v_l) \mapsto v_1 \wedge \cdots \wedge v_l$. Choose a basis $(b^1, \ldots, b^n)$ of V. Then a basis of $\wedge^l(V)$ is given by (b^I) where the multiindex $I = (i_1, \ldots, i_l), 1 \le i_1 < i_2 \cdots < i_l \le n$ runs over all ordered l-tuples in lexicographical order. We call this the *standard basis of* $\wedge^l(V)$. Thus, dimension of $\wedge^l(V)$ is $\binom{n}{l}$. Later, it will be convenient to allow complex coefficients, but for now all vector spaces are real. Now let $f : V \to V$ be a linear mapping, given with regard to the basis $(b^1, \ldots, b^n)$ by the matrix $A = (a_{ij})$. Then the mapping $V \times \cdots \times V \to \wedge^l(V)$ given by

$$(v_1, \ldots, v_l) \mapsto Av_1 \wedge \cdots \wedge Av_l$$

is multilinear and alternating and therefore, by the universal property of $\wedge^l(V)$, must factor through $\wedge^l(V)$. The induced linear mapping $\wedge^l(f) : \wedge^l(V) \to \wedge^l(V)$ given in matrix form for monomials as

$$A_{\#}(v_1 \wedge \cdots \wedge v_l) := Av_1 \wedge \cdots \wedge Av_l$$

is the *l-th exterior power* of A. For an ordered l-tuple $J = (j_1, \ldots, j_l)$ we have

$$\begin{aligned} A_{\#}(b^J) &= Ab^{j_1} \wedge Ab^{j_2} \wedge \cdots \wedge Ab^{j_l} \\ &= \left(\sum_{i_1=1}^{n} a_{i_1 j_1} b^{i_1} \right) \wedge \cdots \wedge \left(\sum_{i_l=1}^{n} a_{i_l j_l} b^{i_l} \right) \\ &= \sum_I A_{IJ} b^I \end{aligned}$$

where A_{IJ} is the determinant of the $l \times l$-minor of A obtained by deleting all i-th rows with $i \notin I$ and all j-th columns with $j \notin J$. The mapping $A_{\#}$ is given by the $\binom{n}{l} \times \binom{n}{l}$ matrix of $l \times l$-minors: $A_{\#} = (A_{IJ})$.
A bilinear form g extends in a natural way to a bilinear form $g_{\#}$ on $\wedge^l(V)$: for monomials $b^I = b^{i_1} \wedge \cdots \wedge b^{i_l}$ and $b^J = b^{j_1} \wedge \cdots \wedge b^{j_l}$ set

$$g(b^I, b^J) = \det(g(b^{i_k}, b^{j_l})_{k,l}) \tag{32}$$

and extend $g_{\#}$ bilinearly to all of $\wedge^l(V)$.

Lemma 4.1 *The following rules apply:*

1. $(AB)_\# = A_\# B_\#$
2. $(A^{-1})_\# = (A_\#)^{-1} = A_\#^{-1}$ *if A is invertible*
3. $(A^T)_\# = (A_\#)^T = A_\#^T$
4. $A_\#(\omega \wedge \eta) = (A_\# \omega) \wedge (A_\# \eta)$

If A is symmetric, orthogonal, diagonal or invertible, then $A_\#$ has the same property.

See [17, 27] for more calculations.

4.2 Exterior Powers of Symplectic Mappings

Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectic linear mapping. It satisfies

$$A^T J A = J, \qquad J = \begin{pmatrix} & -I \\ +I & \end{pmatrix}.$$

For $A_\# : \wedge^l(\mathbb{R}^{2n}) \to \wedge^l(\mathbb{R}^{2n})$ the functorial properties imply

$$A_\#^T J_\# A_\# = J_\#$$

and therefore $A_\#$ preserves the bilinear form associated to $J_\#$. Because of

$$J_\#^T J_\# = I$$

we know that $J_\#$ is invertible and describes a non-degenerate bilinear form. The equation above shows that $J_\#$ is orthogonal; but for even l it is not symplectic anymore since skew-symmetry of J,

$$J^T = -J$$

goes over to

$$J_\#^T = (-1)^l J_\#.$$

Example. $\wedge^2(\mathbb{R}^4)$
Let $V = \mathbb{R}^4$ with the standard basis (e^1, e^2, e^3, e^4). We calculate $J_\# = \wedge^2(J)$.
Obviously $Je^1 = e^3$, $Je^2 = e^4$, $Je^3 = -e^1$, $Je^4 = -e^2$. The standard basis of $\wedge^2(\mathbb{R}^4)$ is $(e^1 \wedge e^2, e^1 \wedge e^3, e^1 \wedge e^4, e^2 \wedge e^3, e^2 \wedge e^4, e^3 \wedge e^4)$. Since $l = 2$, $J_\#$ will be symmetric. A calculation gives (note: $J_\# e^1 \wedge e^3 = Je^1 \wedge Je^3 = e^3 \wedge (-e^1) = e^1 \wedge e^3$)

$$J_\# = \begin{pmatrix} & & & & & 1 \\ & 1 & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & 1 & \\ 1 & & & & & \end{pmatrix}.$$

This symmetric matrix (and the associated bilinear form) is orthogonalizable: choose a new basis (w^i) with

$$\begin{aligned}
w^1 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^2 + e^3 \wedge e^4) \\
w^2 &= e^1 \wedge e^3 \\
w^3 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3) \\
w^4 &= e^2 \wedge e^4 \\
w^5 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4) \\
w^6 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^4 - e^2 \wedge e^3).
\end{aligned}$$

The orthogonal matrix of the change of base is

$$M = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix}$$

and we get

$$M^T J_{\#} M = \begin{pmatrix}
1 & & & & & \\
& 1 & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& & & & -1 & \\
& & & & & -1
\end{pmatrix}.$$

Let us discuss the example $\wedge^2(\mathbb{R}^4)$ further.
The symmetric bilinear form represented by $J_{\#}$ is of type $(4,2)$. We can decompose $\wedge^2(\mathbb{R}^4)$ into subspaces

$$\wedge^2(\mathbb{R}^4) = V^+ \oplus V^- \qquad V^+ = \mathrm{Span}\{w^1, w^2, w^3, w^4\} \qquad V^- = \mathrm{Span}\{w^5, w^6\} \tag{33}$$

on which $J_{\#}$ is positive and negative definite (these are the eigenspaces of the linear operator $J_{\#}$). This decomposition is not invariant under symplectic maps. The Cartan decomposition $A = U_1 D U_2$ of A gives $A_{\#} = U_{1\#} D_{\#} U_{2\#}$ where $D_{\#}$ is still diagonal and $U_{i\#}$ are unitary,

$$U_{i\#}^T J_{\#} U_{i\#} = J_{\#} \tag{34}$$

$$U_{\#}^T U_{\#} = I. \tag{35}$$

Equation (34) just expresses the invariance of the bilinear form $J_\#$ under $U_{i\#}$. Suppose v and w are the coordinates of a vector relative to the standard basis. Then

$$(U_\# v)^T J_\# (U_\# w) = v^T U_\# J_\# U_\# v = v^T J_\# w.$$

Equations (34) and (35) together give $U_{i\#} J_\# = J_\# U_{i\#}$. Therefore $U_{i\#}$ preserves the eigenspaces of $J_\#$. This allows us to decompose $U_{i\#} = U_i^1 \oplus U_i^2$ into $U_i^1 \in Gl(V^+)$, $U_i^2 \in Gl(V^-)$, $i = 1, 2$, and we have

$$\begin{pmatrix} U_i^1 & 0 \\ 0 & U_i^2 \end{pmatrix}^T \begin{pmatrix} U_i^1 & 0 \\ 0 & U_i^2 \end{pmatrix} = \begin{pmatrix} U_i^{1T} U_i^2 & 0 \\ 0 & U_i^{2T} U_i^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Therefore in fact $U_{i\#} \in O(4) \oplus O(2)$ with determinant 1.
Since $D_\#$ does not commute with $J_\#$ it does not preserve the decomposition. From $D = diag(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1})$ we get

$$D_\# = \begin{pmatrix} \lambda_1\lambda_2 & & & & & \\ & 1 & & & & \\ & & \lambda_1\lambda_2^{-1} & & & \\ & & & \lambda_1^{-1}\lambda_2 & & \\ & & & & 1 & \\ & & & & & (\lambda_1\lambda_2)^{-1} \end{pmatrix}.$$

We have not yet used the equation $A^*\omega_0 = \omega_0$ in the dual space $\wedge^l(\mathbb{R}^{4*})$ of 2-forms; $\omega_0 = de^1 \wedge de^3 + de^2 \wedge de^4$ is the standard symplectic form. Using the duality isomorphism $\mathbb{R}^{4*} \to \mathbb{R}^4$, $de^i \mapsto e^i$, and taking exterior powers we get

$$A_\#(e^1 \wedge e^3 + e^2 \wedge e^4) = e^1 \wedge e^3 + e^2 \wedge e^4$$

(the matrix $A_\#$ is the transpose of A^*). Make a orthogonal coordinate change N on V^+ so that $v_0 := 1/\sqrt{2}(e^1 \wedge e^3 + e^2 \wedge e^4)$ is the first basis vector. Then the first column of $\tilde{A}_\# = N^T A_\# N$ becomes $(1, 0, 0, 0, 0, 0)$. Since the transpose of a symplectic matrix A is symplectic too, the first row of $\tilde{A}_\#$ is also $(1, 0, 0, 0, 0, 0)$. Hence

$$\tilde{A}_\# = \left(\begin{array}{c|c} 1 & \\ \hline & * \end{array}\right).$$

and $U_{i\#}$ take the form

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & O(3) & \\ \hline & & O(2) \end{array}\right).$$

4.3 A symplectic Hodge Operator

Many of the results of [17] are centered around Hodge theory and the related operators. For instance, the non-conformality of a mapping is detected in the distortion of the eigenspace decomposition of the Hodge $*$–operator. This led to the question whether there is a symplectic analogue. The usual $*$–operator is defined with the aid of a non-degenerate bilinear form (the Riemannian metric); this construction can be imitated with the symplectic form. The following presentation of the *symplectic Hodge operator* is mainly taken from [5].
Let (V, ω) be a $2n$-dimensional symplectic vector space with basis $(b^1, \ldots, b^{2n})$. The symplectic form is a non-degenerate bilinear form and defines therefore a duality isomorphism $\Psi : V^* \to V$ such that

$$\omega(v, \Psi(\alpha)) = \alpha(v) \qquad \forall v \in V, \alpha \in V^*.$$

This isomorphism extends to an isomorphism

$$\wedge^2(\Psi) : \wedge^2(V^*) \to \wedge^2(V).$$

The antisymmetric tensor

$$G := - \wedge^2 (\Psi)(\omega)$$

can be interpretated as antisymmetric bilinear form

$$G : V^* \times V^* \to \mathbb{R}.$$

Its exterior powers $\wedge^k G : \wedge^k(V^*) \times \wedge^k(V^*) \to \mathbb{R}$ are $(-1)^k$-symmetric.

Example. Consider $\mathbb{R}^{2n} = \{(x_1 \ldots, x_n, y_1 \ldots, y_n)\}$ with standard basis $(e^1, \ldots, e^{2n})$ and standard symplectic form $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$. Then $\Psi(dy^i) = -e^i$, $\Psi(dx^i) = e^{n+i}$ because of

$$\omega\left(e^{n+i}, \Psi(v)\right) = \begin{cases} 1 & v = dy^i \\ 0 & v \neq dy^i \end{cases} .$$

With

$$\wedge^2\Psi(dx^i \wedge dy^i) = e^{n+i} \wedge \left(-e^i\right) = e^i \wedge e^{n+i}$$

we get

$$G = \sum_{i=1}^n e^{n+i} \wedge e^i.$$

Thus formally G is the negative of the dual of ω_0. But logically, the bilinear mapping G should be distinguished from the bivector $e^1 \wedge e^3 + e^2 \wedge e^4 \in \wedge^2(\mathbb{R}^4)$.

□

The symplectic Hodge operator $*_\omega$ on forms is now given by the condition

$$\begin{aligned} \beta \wedge (*_\omega \alpha) &= \wedge^k G(\beta, \alpha) \cdot \mathrm{Vol}_V \qquad \forall \beta \in \wedge^k(V^*) \\ *_\omega : \wedge^k(V^*) &\to \wedge^{2n-k}(V^*). \end{aligned}$$

where the volume form Vol_V is defined as $\mathrm{Vol}_V := \frac{1}{n!} \wedge^l \omega$.
By duality, we can transfer it to $\wedge^k(V)$ where it is given by $(\mathrm{Vol}_V^* := \frac{1}{n!} \wedge^l G)$

$$\begin{aligned} w \wedge (*_\omega v) &= \wedge^k \omega(w, v) \cdot \mathrm{Vol}_V^* \\ *_\omega : \wedge^k(V) &\to \wedge^{2n-k}(V). \end{aligned}$$

Lemma 4.2 *[5]*

1. *For $\alpha \in \wedge^k(V^*)$ we have $*_\omega(*_\omega \alpha) = \alpha$.*
2. *For $\alpha, \beta \in \wedge^k(V^*)$ we have $\beta \wedge (*_\omega \alpha) = (-1)^k \alpha \wedge (*_\omega \beta)$.*

We reformulate our considerations in the context of symplectic manifolds. For the more general case of Poisson manifolds see [5], [44]. So let (M, ω) be a smooth symplectic manifold of dimension $2n$. We already know G, the antisymmetric tensor of order 2, and the bundle isomorphism $\Psi : T^*M \to TM$. The pairing

$$G : T^*M \times T^*M \to C^\infty(M)$$

extends to

$$\wedge^k G : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M).$$

where $\Omega^k(M) = \wedge^k(T^*M)$ is the space of smooth differential k-forms. The symplectic $*_\omega$-operator on differential forms is given by

$$\begin{aligned} \beta \wedge (*_\omega \alpha) &= \wedge^k G(\beta, \alpha) \cdot \mathrm{Vol}_M \\ *_\omega : \Omega^k(M) &\to \Omega^{2n-k}(M) \end{aligned}$$

and lemma 4.2 holds. In the general context of Poisson manifolds we can compare the $*_\omega$-operator with the Koszul differential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ which is similar to the codifferential. They are linked by the formula $\delta = (-1)^{k+1} *_\omega d *_\omega$ on $\Omega^k(M)$. For comparing $*_\omega$ with the usual Hodge $*$-operator we confine ourselves to the case of Kähler manifolds. So let M be a complex n-dimensional manifold with hermitian metric $h = g + i \cdot \omega$. The imaginary part is given by a non-degenerate real 2-form ω which is symplectic since M is Kähler. The real part g is a Riemannian metric and comes with the usual $*$-operator

$$\begin{aligned} \beta \wedge (*_g \alpha) &= \wedge^k g^{-1}(\beta, \alpha) \cdot \mathrm{Vol}_g \qquad \mathrm{Vol}_g = \mathrm{Vol}_M \\ *_g : \Omega^k(M) &\to \Omega^{2n-k}(M). \end{aligned}$$

Here g^{-1} denotes the dual of g (the matrix representation of g^{-1} in the dual basis is the inverse of g). It satisfies

$$g^{-1}(\beta, \alpha) = g(\Phi(\beta), \Phi(\alpha))$$

with the duality isomorphism

$$\begin{aligned} \Phi : T^*M &\to TM \\ \alpha(v) &= g(v, \Phi(\alpha)) \qquad \forall v \in TM, \alpha \in T^*M. \end{aligned}$$

The operators $*_\omega$and $*_g$ agree up to powers of i on complex forms of bidegree (p,q):

Theorem 4.3 *[5] For $\alpha \in \Omega^{p,q}(M)$ we have*

$$*_\omega \alpha = i^{p-q} *_g \alpha.$$

Example. We calculate the symplectic Hodge operator $*_\omega$ related to the standard symplectic form ω_0 on the 6-dimensional space of 2-forms on $\mathbb{R}^4$:

$$*_\omega : \wedge^2(\mathbb{R}^{4*}) \to \wedge^2(\mathbb{R}^{4*}).$$

Since the above mapping is idempotent ($*_\omega *_\omega = I$), the space $\wedge^2(\mathbb{R}^{4*})$ decomposes as a direct sum of its eigenspaces to eigenvalues $+1$ and -1

$$\wedge^2(\mathbb{R}^{*4}) = \wedge^+_\omega \oplus \wedge^-_\omega \tag{36}$$

where

$$\wedge^+_\omega := \left\{\alpha \in \wedge^2(\mathbb{R}^{4*}) : *_\omega \alpha = \alpha\right\}$$

is the space of self-dual forms and

$$\wedge^-_\omega := \left\{\alpha \in \wedge^2(\mathbb{R}^{4*}) : *_\omega \alpha = -\alpha\right\}$$

is the space of anti-self-dual forms.

The calculation of the action of $*_\omega$ on 2-forms is simplified by the fact that locally a symplectic manifolds always looks like a product of symplectic manifolds ([5, p. 100]): if (M^{2n_1}, ω_1) and (M^{2n_2}, ω_2) are symplectic manifolds and $\alpha_i \in \wedge^{k_i}(M_i)$ are given, then $\alpha_1 \wedge \alpha_2 \in \wedge^{k_1+k_2}(M_1 \times M_2)$ and

$$*_\omega(\alpha_1 \wedge \alpha_2) = (-1)^{k_1 k_2}(*_{\omega_1}\alpha_1) \wedge (*_{\omega_2}\alpha_2) = (*_{\omega_2}\alpha_2) \wedge (*_{\omega_1}\alpha_1)$$

where $\omega = \omega_1 + \omega_2$. For $\dim(M) = 2$ and $\omega = dx \wedge dy$ the following rules hold:

$$\begin{aligned}
*_\omega f &= f dx \wedge dy \qquad f \in C^\infty(M) \\
*_\omega \alpha &= -\alpha \qquad \alpha \in \Omega^1(M) \\
*_\omega(f dx \wedge dy) &= f \qquad f \in C^\infty(M).
\end{aligned}$$

We decompose $(\mathbb{R}^4, \omega_0) = (\mathbb{R}^2, dx^1 \wedge dy^1) \times (\mathbb{R}^2, dx^2 \wedge dy^2)$ and calculate the effect on 2-forms. We use the abbreviations $*_i = *_{\omega_i}$ and $\omega_i = dx^i \wedge dy^i$.

$$\begin{aligned}
*_\omega(dx^1 \wedge dy^2) &= (*_2 dy^2) \wedge (*_1 dx^1) = (-dy^2) \wedge (-dx^1) = -dx^1 \wedge dy^2 \\
*_\omega(dx^1 \wedge dx^2) &= (*_2 dx^2) \wedge (*_1 dx^1) = (-dx^2) \wedge (-dx^1) = -dx^1 \wedge dx^2 \\
*_\omega(dx^1 \wedge dy^1) &= *_\omega((dx^1 \wedge dy^1) \wedge 1) = (*_2 1) \wedge (*_1(dx^1 \wedge dy^1)) = \omega_2 \cdot 1 = dx^2 \wedge dy^2 \\
*_\omega(dx^2 \wedge dy^2) &= *_\omega(1 \wedge (dx^2 \wedge dy^2)) = (*_2(dx^2 \wedge dy^2)) \wedge (*_1 1) = 1 \cdot \omega_1 = dx^1 \wedge dy^1 \\
*_\omega(dy^1 \wedge dx^2) &= (*_2 dx^2) \wedge (*_1 dy^1) = (-dx^2) \wedge (-dy^1) = -dy^1 \wedge dx^2 \\
*_\omega(dy^1 \wedge dy^2) &= (*_2 dy^2) \wedge (*_1 dy^1) = (-dy^2) \wedge (-dy^1) = -dy^1 \wedge dy^2.
\end{aligned}$$

The eigenspaces are therefore ($\tilde{\omega_0} := dx^1 \wedge dy^1 - \ dx^2 \wedge dy^2$)

$$\begin{aligned} \wedge^+_\omega &= \left\langle dx^1 \wedge dy^1 + \ dx^2 \wedge dy^2 \right\rangle = \langle \omega_0 \rangle & (37) \\ \wedge^-_\omega &= \left\langle \tilde{\omega_0}, dx^1 \wedge dx^2,\ dy^1 \wedge dy^2,\ dx^1 \wedge dy^2,\ dy^1 \wedge dx^2 \right\rangle . & (38) \end{aligned}$$

We check the claim $*_\omega(dx^1 \wedge dy^1) = dx^2 \wedge dy^2$ using the definition of the symplectic star operator. First we calculate $\wedge^2 G(\beta, \alpha)$ for $\alpha = \alpha_1 \wedge \alpha_2$, $\alpha_1 = dx^1$, $\alpha_2 = dy^1$, $\beta = \beta_1 \wedge \beta_2$, $\beta_i = \beta_i^1 dx^1 + \beta_i^2 dx^2 + \beta_i^3 dy^1 + \beta_i^4 dy^2$. Using the results of the previous example,

$$\begin{aligned} G(\beta_i, \alpha_1) &= G(\beta_i, dx^1) = e^3(\beta_i) \cdot 1 = \beta_i^3 \\ G(\beta_i, \alpha_2) &= -e^1(\beta_i) \cdot 1 = -\beta_i^1 \end{aligned}$$

we get

$$\wedge^2 G(\beta, \alpha) = \det \begin{pmatrix} \beta_1^3 & -\beta_1^1 \\ \beta_2^3 & -\beta_2^1 \end{pmatrix} = \beta_1^1 \cdot \beta_2^3 - \beta_2^1 \cdot \beta_1^3.$$

Compare this with

$$\begin{aligned} \beta \wedge (dx^2 \wedge dy^2) &= (\beta_1^1 dx^1 + \beta_1^2 dx^2 + \beta_1^3 dy^1 + \beta_1^4 dy^2) \wedge (\beta_2^1 dx^1 + \beta_2^2 dx^2 + \beta_2^3 dy^1 + \beta_2^4 dy^2) \wedge \\ &\quad \wedge dx^2 \wedge dy^2 \\ &= (\beta_1^1 dx^1 + \beta_1^3 dy^1) \wedge (\beta_2^1 dx^1 + \beta_2^3 dy^1) \wedge dx^2 \wedge dy^2 \\ &= (\beta_1^1 \cdot \beta_2^3 - \beta_2^1 \cdot \beta_1^3) \cdot dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2. \end{aligned}$$

This agrees with the definition $\beta \wedge (*_\omega \alpha) = \wedge^2 G(\beta, \alpha) \frac{\omega_0 \wedge \omega_0}{2}$. □

Lemma 4.4 *Let* $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. *Then*

1. $\wedge^k \omega_0(A_\# v, w) = \wedge^k \omega_0(v, J_\#^T A_\#^T J_\# w) \qquad v, w \in \wedge^k \mathbb{R}^{2n}$
2. $\wedge^k G(A^* \alpha, \beta) = \wedge^k G(\alpha, J^* A^{*T} J^{*T} \beta) \qquad \alpha, \beta \in \wedge^k(\mathbb{R}^{2n*})$

Proof: For 1) take $v = v_1 \wedge \dots \wedge v_k$ and $w = w_1 \wedge \dots \wedge w_k$ in $\wedge^k(\mathbb{R}^{2n})$. The statement follows from

$$\omega_0(Av_i, w_j) = \langle Av_i, Jw_j \rangle = \left\langle v_i, A^T J w_j \right\rangle = \omega_0(v_i, J^T A^T J w_j)$$

and definition (32):

$$\wedge^k \omega_0(A_\# v, w) = \det(\omega_0(Av_i, w_j)_{i,j}) = \wedge^k \omega_0(v, J_\#^T A_\#^T J_\# w).$$

For 2), which is the dual to 1), note that the usual pull-back $A^* \alpha$ of a form α is described in the dual basis by the transpose of A. In particular, $J^* dx^i = -dy^i$, $J^* dy^1 = dx^i$. In our notation $A^* = A_\#^T$. We reserve the $\#$–notation for the induced mapping on vectors and the $*$–notation for the forms. Take k-forms $\alpha = \alpha_i \wedge \dots \wedge \alpha_k$, $\beta = \beta_1 \wedge \dots \wedge \beta_k$ in $\wedge^k(\mathbb{R}^{2n*})$. Verify

$$\begin{aligned} G(A^* \alpha_i, \beta_j) &= \left\langle A^* \alpha_i, J^{*T} \beta_j \right\rangle = \left\langle \alpha_i, A^{*T} J^{*T} \beta_j \right\rangle \\ &= G(\alpha_i, J^* A^{*T} J^{*T} \beta_j) = G(\alpha_i, (J^T A^T J)^* \beta_j). \end{aligned}$$

□

Lemma 4.5 *For any matrix $A \in Gl(\mathbb{R}^{2n})$*

$$J_\#^T A_\#^T J_\# *_\omega A_\# = (det A) *_\omega : \wedge^k(\mathbb{R}^{2n}) \to \wedge^{2n-k}(\mathbb{R}^{2n}). \tag{39}$$

Proof: 1) Choose $\alpha \in \wedge^k \mathbb{R}^{2n}, \beta \in \wedge^{2n-k}\mathbb{R}^{2n}$. Using the previous lemma and $*_\omega *_\omega = I$ we compute

$$\begin{aligned}
\wedge^k \omega_0(\beta, J_\#^T A_\#^T J_\# *_\omega A_\# \alpha) \cdot \mathrm{Vol}^*_{\mathbb{R}^{2n}} &= \wedge^k \omega_0(A_\# \beta, *_\omega A_\# \alpha) \cdot \mathrm{Vol}^*_{\mathbb{R}^{2n}} \\
&= A_\# \beta \wedge *_\omega *_\omega A_\# \alpha \\
&= A_\# \beta \wedge A_\# *_\omega *_\omega \alpha = A_\#(\beta \wedge *_\omega *_\omega \alpha) \\
&= A_\# \wedge^k \omega_0(\beta, *_\omega \alpha) \cdot \mathrm{Vol}^*_{\mathbb{R}^{2n}} \\
&= (\det A) \wedge^k \omega_0(\beta, *_\omega \alpha) \cdot \mathrm{Vol}^*_{\mathbb{R}^{2n}}
\end{aligned}$$

The relation $J_\#^T A_\#^T J_\# *_\omega = (det A) *_\omega$ follows since $\wedge^k \omega_0$ is non-degenerate. □

Corollary 4.6 *The decomposition $\wedge^k(\mathbb{R}^{2n}) = \wedge^+_\omega \oplus \wedge^-_\omega$ is invariant under symplectic mappings.*

Proof: If A is symplectic, then $J_\#^T A_\#^T J_\# = A_\#^{-1}$. Lemma 4.5 gives $A_\#^{-1} *_\omega A_\# = *_\omega$, or equivalently $A *_\omega = *_\omega A$. The claim follows. □

4.4 Decompositions of $\wedge(\mathbb{R}^4)$

We have already taken note of the decomposition of $\wedge^2(\mathbb{R}^{4*})$ into eigenspaces of the symplectic $*_\omega$–operator (36-38)

$$\wedge^2(\mathbb{R}^{4*}) = \wedge^+_\omega \oplus \wedge^-_\omega = \langle\omega_0\rangle \oplus \left\langle \tilde{\omega}_0,\ dx^1\wedge dx^2,\ dy^1\wedge dy^2,\ dx^1\wedge dy^2,\ dy^1\wedge dx^2\right\rangle .$$

We shall introduce two more decompositions. Namely, into eigenspaces of the Euclidean Hodge $*$–operator and also into eigenspaces of $J_\# = \wedge^2(J)$. All these operators have eigenvalues $+1$ and -1. Our aim is to compare the decompositions and to derive a Beltrami equation relative to $J_\#$. It will be easier to work with the complexified spaces $\wedge^2_{\mathbb{C}}(\mathbb{R}^4)$ and $\wedge^2_{\mathbb{C}}(\mathbb{R}^{4*})$. In complex notation we have

$$\omega_0 = \frac{i}{2}(dz^1\wedge d\bar{z}^1 + dz^2\wedge d\bar{z}^2), \qquad \tilde{\omega}_0 = \frac{i}{2}(dz^1\wedge d\bar{z}^1 - dz^2\wedge d\bar{z}^2)$$

and

$$\wedge^2_{\mathbb{C}}(\mathbb{R}^{4*}) = \langle\omega_0\rangle \oplus \left\langle \tilde{\omega}_0, dz^1\wedge dz^2, dz^1\wedge d\bar{z}^2, dz^2\wedge d\bar{z}^1, d\bar{z}^1\wedge d\bar{z}^2\right\rangle . \tag{40}$$

The real Hodge decomposition $\wedge^2_{\mathbb{C}}(\mathbb{R}^{4*}) = \wedge^+ \oplus \wedge^- = \{\alpha + *\alpha\} \oplus \{\alpha - *\alpha\}$ is

$$\begin{aligned}
\wedge^+ &= \left\langle dx^1\wedge dx^2 + dy^1\wedge dy^2, dx^1\wedge dy^1 - dx^2\wedge dy^2, dx^1\wedge dy^2 + dx^2\wedge dy^1\right\rangle \\
\wedge^- &= \left\langle dx^1\wedge dx^2 - dy^1\wedge dy^2, dx^1\wedge dy^1 + dx^2\wedge dy^2, dx^1\wedge dy^2 - dx^2\wedge dy^1\right\rangle .
\end{aligned}$$

With $dz^i\wedge dz^j = dx^i\wedge dx^j - dy^i\wedge dy^j + i(dx^i\wedge dy^j - dx^j\wedge dx^i)$ and $dz^i\wedge d\bar{z}^i = -2i dx^i\wedge dy^i$ we get

$$\begin{aligned}
\wedge^2_{\mathbb{C}}(\mathbb{R}^{4*}) &= \wedge^+ \oplus \wedge^- = \left\langle dz^1\wedge d\bar{z}^2, d\bar{z}^1\wedge dz^2, \tilde{\omega}_0\right\rangle \oplus \left\langle dz^1\wedge dz^2, d\bar{z}^1\wedge d\bar{z}^2, \omega_0\right\rangle \\
&= \wedge^+ \oplus \wedge^-_{red} \oplus \langle\omega_0\rangle
\end{aligned}$$

where $\wedge^-_{red} := \langle dz^1\wedge dz^2, d\bar{z}^1\wedge d\bar{z}^2\rangle$.
Decomposing $\wedge^2_{\mathbb{C}}(\mathbb{R}^{4*}) = \wedge^+_J \oplus \wedge^-_J$ relative to $J_\#$ is just sorting according to complex type $T_{p,q}$ (since $Jdz^j = idz^j$ and $Jd\bar{z}^j = -id\bar{z}^j$):

$$\begin{aligned}
\wedge^+_J &:= \{\alpha + J_\#\alpha\} = T_{1,1} = \wedge^+ \oplus \langle\omega_0\rangle \\
\wedge^-_J &:= \{\alpha - J_\#\alpha\} = T_{2,0} \oplus\ T_{0,2} = \wedge^-_{red}.
\end{aligned} \tag{41}$$

With the help of the standard dualization $z^i \mapsto dz^i, \bar{z}^i \mapsto d\bar{z}^i$ we decompose $\wedge^2_{\mathbb{C}}(\mathbb{R}^4)$ accordingly (with the same notations); for instance

$$\begin{aligned}
\wedge^+ &= \left\langle Z_1\wedge\overline{Z}_2, \overline{Z}_1\wedge Z_2, Z_1\wedge\overline{Z}_1 - Z_2\wedge\overline{Z}_2\right\rangle \\
\wedge^-_{red} &= \left\langle Z_1\wedge Z_2, \overline{Z}_1\wedge\overline{Z}_2\right\rangle = T^{2,0}\oplus T^{0,2}.
\end{aligned}$$

A linear symplectic mapping satisfies $A^*\omega_0 = \omega_0$ and $A_\# v_0 = v_0$ where $v_0 := \frac{i}{2}(Z_1 \wedge \overline{Z}_1 + Z_2 \wedge \overline{Z}_2)$ is the dual of the symplectic form ω_0. Corollary 4.6 and (37), (41) show that $\langle v_0 \rangle$ and its complement $\wedge^+ \oplus T_{2,0} \oplus T_{0,2} = \wedge^+ \oplus \wedge^-_{red}$ are invariant (see also the reasoning at the end of section 4.2). Thus we can split off $\langle v_0 \rangle$ and look at

$$A_\# : \wedge^+ \oplus \wedge^-_{red} \to \wedge^+ \oplus \wedge^-_{red}.$$

This decomposition is invariant under holomorphic mappings. For general symplectic mappings the distortion of the spaces is measured by a mapping $\nu : \wedge^+ \to \wedge^-_{red}$ which we shall now describe.
Apply the definition of the complex dilatation μ to $f_* : \wedge^+ \to \wedge^+ \oplus \wedge^-_{red}$ and decompose the image of $\wedge^+$:

$$\begin{aligned} f_*(Z_i \wedge \overline{Z}_j) &= (V_i + \overline{W}_i) \wedge (\overline{V}_j + W_j) = (V_i \wedge \overline{V}_j + \overline{W}_i \wedge W_j) + (V_i \wedge W_j + \overline{W}_i \wedge \overline{V}_j) \\ &= (V_i \wedge \overline{V}_j + \overline{\mu V_i} \wedge \mu V_j) + (V_i \wedge \mu V_j + \overline{\mu V_i} \wedge \overline{V}_j). \end{aligned}$$

With the $\mathbb{R}$-linear operators (c is complex conjugation)

$$\begin{aligned} I \otimes \mu \circ c &: Z \wedge Z' \mapsto Z \wedge \mu\overline{Z'} \\ c \circ \mu \otimes I &: Z \wedge Z' \mapsto \overline{\mu Z} \wedge Z' \\ c \circ \mu \otimes \mu \circ c &: Z \wedge Z' \mapsto \overline{\mu Z} \wedge \mu\overline{Z'} \end{aligned}$$

we write

$$f_*(Z_i \wedge \overline{Z}_j) = (I + c \circ \mu \otimes \mu \circ c) V_i \wedge \overline{V}_j + (I \otimes \mu \circ c + c \circ \mu \otimes I) V_i \wedge \overline{V}_j.$$

The second vector above is of pure type $(2,0)$ or $(0,2)$ and thus is in $\wedge^-_{red}$. The first vector is in $\wedge^+$; since we sorted according to complex type it lies a priori in the 4-dimensional space $\wedge^{1,1}$ of vectors with mixed type. But it does not contain any component in the invariant direction $\langle v_0 \rangle = \langle Z_1 \wedge \overline{Z}_1 + Z_2 \wedge \overline{Z}_2 \rangle$ and so it is contained in the 3-dimensional subspace $\wedge^+ \subset T_{1,1}$.
The projection $\pi_{\wedge^- red}$ can be calculated from the projection $\pi_{\wedge^+}$ since the operator $I + c \circ \mu \otimes c \circ \bar{\mu}$ is invertible because of $||\mu||_\infty < 1$. In this way we arrive at the following Beltrami-like equation with a "complex dilation" $\nu : \wedge^+ \to \wedge^-_{red}$:

$$\begin{aligned} \pi_{\wedge^-_{red}} \circ f_* v &= \nu(\pi_{\wedge^+} \circ f_* v) \qquad v \in \wedge^+ \\ \nu &= \frac{I \otimes \mu \circ c + c \circ \mu \otimes I}{I + c \circ \mu \otimes \mu \circ c}. \end{aligned}$$

The notation as a fraction is unambiguous since the operators commute.

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